

Supergravity Black Holes and Billiards and Liouville integrable structure of dual Borel algebras[†]

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Abstract

In this paper we show that the supergravity equations describing both cosmic billiards and a large class of black-holes are, generically, both Liouville integrable as a consequence of the same universal mechanism. This latter is provided by the Liouville integrable Poissonian structure existing on the dual Borel algebra \mathbb{B}_N of the simple Lie algebra A_{N-1} . As a by product we derive the explicit integration algorithm associated with all symmetric spaces U/H^* relevant to the description of time-like and space-like p -branes. The most important consequence of our approach is the explicit construction of a complete set of conserved involutive hamiltonians $\{\mathfrak{h}_\alpha\}$ that are responsible for integrability and provide a new tool to classify flows and orbits. We believe that these will prove a very important new tool in the analysis of supergravity black holes and billiards.

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1 Introduction

Explicit supergravity solutions of pure and matter-coupled supergravity in diverse dimensions play an important role in the study of solitonic and instantonic states of superstring theory, in particular p -brane states [1, 2].

Indeed one large, diversified and important class of supergravity solutions¹ is provided by the p -brane ones that are divided in two subclasses:

- The space-like p -brane solutions that have an Euclidian world-volume and are time-dependent, all fields being functions of the time parameter t .
- The time-like p -brane solutions that have a Minkowskian world volume and are stationary, the fields depending on another parameter t , typically measuring the distance from the brane.

A view-point independently introduced in [4] and [5], and systematically developed in [6], [7], [8], [9], [10], identified, at least for space-branes, the field equations of supergravity corresponding to such solutions with the geodesic equations on the corresponding moduli space that is mostly a homogeneous space and most frequently also a symmetric space U/H . This identification allowed the in-depth study of supergravity cosmic billiards [11], [12], [13], [14], [15] and lead to the discovery of their complete integrability [9], [10].

It was an idea already circulating for some time in the community that also the construction of time-like p -branes, in particular rotational symmetric black-hole solutions, could be reduced to the problem of geodesic motion on appropriate moduli spaces that would, this time, be Lorentzian rather Euclidean coset manifolds U/H^* . This idea found a precise formulation in the recent publication [16].

In connection with these applications, the question of integrability of the differential systems of equations describing geodesic motion on homogeneous spaces and in particular on symmetric non-compact cosets U/H , acquires particular relevance. As we emphasize in section 2, this question is intimately related with the issue of *normed solvable Lie algebras*, namely solvable Lie algebras \mathcal{S} equipped with a non-degenerate norm \langle, \rangle , which is positive definite in the space-brane (=billiard) case and indefinite in the time-brane (=black hole) case.

In this paper, by performing a change of logical reference frame that replaces the route from geometry to Lie algebra into the opposite one and by gluing together pieces of mathematical knowledge dispersed in the literature, we show that:

1. The integrability of all the various homogeneous models, both Euclidian and Lorentzian follows from the Liouville integrability of a universal parent model, associated with the Borel subalgebra \mathbb{B}_N of the A_{N-1} Lie algebra. The integrability of the parent extends to its children algebras \mathcal{S} if the always existing embedding $\mathcal{S} \hookrightarrow \mathbb{B}_N$ is adequate.

¹For a review and for a large set of references see for instance [3].

2. Liouville integrability of \mathbb{B}_N is an intrinsic property of this algebra which allows to construct an adequate number of universal hamiltonians $\{\mathfrak{h}_\alpha\}$ in involution.
3. The norm $<, >$ on any solvable Lie algebra \mathcal{S} is not an independent external datum, rather it is intrinsically defined by the restriction to \mathcal{S} of the unique quadratic hamiltonian \mathfrak{h}_0 on \mathbb{B}_N , once the embedding $\mathcal{S} \hookrightarrow \mathbb{B}_N$ has been defined.
4. All symmetric coset models U/H^* defined as follows have integrable geodesic equations. The Lie algebra \mathbb{U} of the numerator is non-compact and the Lie algebra \mathbb{H}^* of the denominator is any of the real sections contained in \mathbb{U} of the complexification $\mathbb{H}_{\mathbb{C}}$ of $\mathbb{H} \subset \mathbb{U}$, the former being the maximal compact subalgebra of the latter.
5. The explicit integration algorithm has a universal form.

Based on our new view-point we also present a new algorithmic approach to the study of the (eventual) integrability of homogeneous normal spaces that are not symmetric spaces, leaving however the actual use of such an algorithm to future publications.

2 A new view-point from old results

In this section we first summarize the basic facts about the Riemannian or pseudo-Riemannian structures that can be defined on a *normed* solvable Lie algebra. Our goal is that of reviewing the construction of the so named Nomizu connection and of its associated geodesic differential equations. The reason is that we aim at a Copernican Revolution. In this context, the classical route was from Riemannian geometry to Lie algebra theory since the problems that motivated the consideration of such mathematical structures were differential geometric in nature: in particular the geometry of scalar manifolds appearing in supergravity theories. It was very helpful and rewarding to find a translation vocabulary that allowed the reformulation of Riemannian geometry into a purely Lie algebraic setup. Yet, in relation with integrability, this classical route obscures one relevant fact: integrability (when it exists) is an a priori intrinsic property of the solvable Lie algebra. This property is intelligently, yet secretly, utilized by the (pseudo)-Riemannian structures. Hence, following our announced Copernican Revolution, we aim at reverting the route, going from solvable Lie algebra theory to (pseudo)-Riemannian geometry, rather than vice-versa. This change of reference frame will prove very helpful in view of old mathematical results, that were a little bit known in the literature on non-linear science [17], [18], [19] but which had so far completely escaped consideration in the current supergravity and superstring literature.

In the next subsection we prepare our Copernican Revolution with a short review of the Ptolemaic system.

2.1 The Ptolemaic system: Nomizu connection on a normal metric solvable Lie algebra

Let us consider a solvable Lie algebra \mathcal{S} . For instance \mathcal{S} can be the Borel subalgebra of a complex semi-simple Lie algebra $\mathbb{G}_{\mathbb{C}}$, namely²:

$$\mathcal{S} = \text{B}(\mathbb{G}_{\mathbb{C}}) \equiv \text{span} \{ \mathcal{H}_i, E^\alpha \} \quad (2.1)$$

or it can be the solvable Lie algebra canonically associated with the pair made by a real form $\mathbb{G}_{\mathbb{R}}$ of $\mathbb{G}_{\mathbb{C}}$ and by its maximal compact subalgebra $\mathbb{H}_c \subset \mathbb{G}_{\mathbb{R}}$ ³:

$$\mathcal{S} = \text{Solv}(\mathbb{G}_{\mathbb{R}}/\mathbb{H}_c) \quad (2.2)$$

Other relevant choices of the solvable Lie algebra \mathcal{S} can be made among those associated with the classification of homogeneous special geometries that appear in the coupling to matter of supergravity theories with eight supercharges in $D = 5$, $D = 4$ and $D = 3$ dimensions⁴:

$$\mathcal{S} \text{ such that } (\exp[\mathcal{S}], <, >) = \text{SUGRA special Riemannian manifolds} \quad (2.3)$$

The above writing refers to the main point of the Ptolemaic system namely to the notion of *normed metric solvable Lie Algebras*. Following the original viewpoint of Alekseevsky we say that a Riemannian manifold (\mathcal{M}, g) is *normal* if it admits a completely solvable Lie group $\exp[\text{Solv}_{\mathcal{M}}]$ of isometries that acts on the manifold in a simply transitive manner (i.e. for every 2 points in the manifold there is one and only one group element connecting them). The group $\exp[\text{Solv}_{\mathcal{M}}]$ is then generated by a so-called *normal metric Lie algebra*, that is a completely solvable Lie algebra $\text{Solv}_{\mathcal{M}}$ endowed with an Euclidean, positive definite, symmetric form $<, >$. The main tool to classify and study the normal homogeneous spaces is provided by the theorem [33], [34] that states that if a Riemannian

²We recall that given the Cartan-Weyl basis of a complex simple Lie algebra $\mathbb{G}_{\mathbb{C}}$, its Borel subalgebra $\text{B}(\mathbb{G}_{\mathbb{C}})$ is defined as the solvable algebra spanned by all the Cartan generators \mathcal{H}_i and by all the step operators E^α associated with all positive roots $\alpha > 0$.

³We recall that the systematic construction of the solvable Lie algebras associated with non-compact symmetric spaces, pioneered in [20] and then extensively developed in the literature, has played a very important role in addressing, solving and systematizing a large number of supergravity problems associated with black-hole solutions [21], [22], [23], [24], with supergravity gaugings [25], [26] and later also with the issue of cosmic billiards introduced in [11], [12], [13], [14], [15] and developed with the systematic help of the solvable Lie algebra representation of supergravity scalar manifolds in [5], [8], [6], [7], [9], [10], [27].

⁴We recall that the classification of special homogeneous manifolds began with the mathematical work of Alekseevsky in 1975 who posed himself the problem of constructing all quaternionic Kähler manifolds with a transitive solvable group of isometries [28] and then was completed and inserted into the c -map framework [29] of supergravity with the work of de Wit et. al. in [30]. Further studies continued in [31] and for a complete recent discussion of the topic and for all relevant further references we refer the reader to [32].

manifold (\mathcal{M}, g) admits a transitive normal solvable group of isometries $\exp[\text{Solv}_{\mathcal{M}}]$, then it is metrically equivalent to this solvable group manifold

$$\begin{aligned} \mathcal{M} &\simeq \exp[\text{Solv}_{\mathcal{M}}] , \\ g|_{e \in \mathcal{M}} &= \langle, \rangle \end{aligned} \quad (2.4)$$

where \langle, \rangle is the Euclidean metric defined on the normal solvable Lie algebra $\text{Solv}_{\mathcal{M}}$.

The conjecture of Alekseevsky was just restricted to quaternionic Kähler manifolds and implied that any such manifold \mathcal{M} that was also homogeneous and of negative Ricci curvature should be normal, in the sense over mentioned, namely a transitive solvable group of isometries $\exp[\text{Solv}_{\mathcal{M}}]$ should exist, that could be identified with the manifold itself. Note that the actual group of isometries U of \mathcal{M} could be much larger than the solvable group,

$$U \supset \exp[\text{Solv}_{\mathcal{M}}] , \quad (2.5)$$

as it is for instance the case for all symmetric spaces

$$\mathcal{M} = \frac{U}{H} \quad (2.6)$$

yet the solvable normed Lie algebra $(\text{Solv}_{\mathcal{M}}, \langle, \rangle)$ had to exist. The problem of classifying the considered manifolds was turned in this way into the problem of classifying the *normal metric solvable Lie algebras* $(\mathcal{S}, \langle, \rangle)$. Note that in Alekseevsky's case the symmetric form \langle, \rangle was not only required to be positive definite but also quaternionic Kähler. Alekseevsky's conjecture actually applies to more general homogeneous Riemannian manifolds than the quaternionic ones: for instance it applies to all those endowed with a special Kähler geometry or with a real special one as the classification of de Wit et. al. [30] demonstrated. It also applies to the symmetric spaces appearing in the scalar sector of extended supergravities with more than eight supercharges. For all these manifolds there exists the corresponding normal metric algebra $(\mathcal{S}, \langle, \rangle)$, in other words they are *normal*. This happens because they are Einstein manifolds of negative Ricci curvature and, although we are not aware of any formal mathematical statement in this direction, one might make the

Conjecture 2.1 *<< Every homogeneous Einstein manifold \mathcal{M} of negative Ricci curvature is normal, namely there exists a normal metric solvable Lie algebra $(\mathcal{S}, \langle, \rangle)$ such that identifying \mathcal{S} with $\text{Solv}_{\mathcal{M}}$ eq. (2.4) applies. >>*

Proving such a conjecture amounts to proving that for every homogeneous Einstein manifold the group of isometries U , which by hypothesis of homogeneity exists and has a transitive action on the manifold, admits a solvable *simply transitive subgroup*⁵ $\exp[\mathcal{S}] \subset U$. If this is true, in view of the already mentioned theorem the rest follows.

⁵Simply transitive means that each group element has no fixed points.

The key assumption is the negative Ricci curvature. Manifolds of positive Ricci curvature, which are typically compact, are excluded. All compact symmetric spaces are indeed counterexamples. For U/H compact there is no transitive solvable subgroup of U.

The recollection of these well known facts was done in order to emphasize the following point. In the Ptolemaic system that starts from Riemannian geometry and arrives at solvable Lie algebras \mathcal{S} , this latter emerges in conjunction with a well defined metric form $<, >$ defined over it. For instance if we focus on Borel solvable algebras $\mathbb{B}(\mathbb{G}_{\mathbb{C}})$, they are endowed with the following canonical metric:

$$\begin{aligned} < \mathcal{H}_i, \mathcal{H}_j > &= 2 \delta_{ij} , \\ < \mathcal{H}_i, E^\alpha > &= 0 , \\ < E^\alpha, E^\beta > &= \delta_{\alpha\beta} \end{aligned} \quad (2.7)$$

whose normalization is absolute if the generators $\{\mathcal{H}_i, E^\alpha, E^{-\alpha}\}$ of the Weyl-Cartan basis for $\mathbb{G}_{\mathbb{C}}$ have the standard normalization

$$\begin{aligned} [\mathcal{H}_i, \mathcal{H}_j] &= 0 , \\ [\mathcal{H}_i, E^\alpha] &= \alpha_i E^\alpha , \\ [E^\alpha, E^\beta] &= N(\alpha, \beta) E^{\alpha+\beta} \quad \text{if } \alpha + \beta \text{ is a root} \quad ; \quad |N(\alpha, \beta)| = 1 , \\ [E^\alpha, E^{-\alpha}] &= 2 \alpha \cdot \mathcal{H} , \\ \alpha \cdot \beta &= \sum_{i=1}^{rank} \alpha_i \beta_i \quad \Rightarrow \quad \{\mathcal{H}_i\} = \text{orthonormalized basis} . \end{aligned} \quad (2.8)$$

The metric (2.7) is singled out by the relation of the Borel algebra $\mathbb{B}(\mathbb{G}_{\mathbb{C}})$ with one specific Riemannian Einstein manifold of negative Ricci curvature. This latter is

$$\mathcal{M} = \frac{G_{ms}}{H_c} \quad (2.9)$$

where G_{ms} denotes the group generated by the unique real section \mathbb{G}_{ms} of the complex Lie algebra $\mathbb{G}_{\mathbb{C}}$ which is maximally split (or, equivalently maximally non-compact) and H_c denotes the unique maximally compact subgroup of G_{ms} . It turns out that $\mathbb{B}(\mathbb{G}_{\mathbb{C}})$ is just the solvable Lie algebra of this Riemannian manifold \mathcal{M}

$$\mathbb{B}(\mathbb{G}_{\mathbb{C}}) = \text{Solv}_{\mathcal{M}} \quad (2.10)$$

and the metric $<, >$ (2.7) on the solvable Lie algebra $\text{Solv}_{\mathcal{M}}$ is that induced by the unique Einstein Riemannian metric on the corresponding coset $\frac{G_{ms}}{H_c}$ (2.9).

Similarly it happens in all other constructions of the Ptolemaic system. The requirements imposed on the final Riemannian Einstein manifold one wants to construct predetermine the metric $<, >$ on the solvable Lie algebra.

Once the metric form is given, the construction of geometry and of the associated geodesic equations follow uniquely. The issue is just that of calculating the Levi-Civita

connection of the metric g induced on the manifold by the form \langle, \rangle defined on the solvable Lie algebra. One way of describing this Levi–Civita connection is by means of the so called *Nomizu operator* acting on \mathcal{S} . The latter is defined as follows:

$$\begin{aligned} \mathbb{L} &: \mathcal{S} \otimes \mathcal{S} \rightarrow \mathcal{S}, \\ \forall X, Y, Z \in \mathcal{S} &: 2 \langle \mathbb{L}_X Y, Z \rangle = \langle [X, Y], Z \rangle - \langle X, [Y, Z] \rangle - \langle Y, [X, Z] \rangle. \end{aligned} \quad (2.11)$$

The *Riemann curvature operator* on \mathcal{S} can be expressed as

$$\text{Riem}(X, Y) = [\mathbb{L}_X, \mathbb{L}_Y] - \mathbb{L}_{[X, Y]}. \quad (2.12)$$

If we introduce a basis of generators $\{T_A\}$ for \mathcal{S} and the corresponding structure constants defined by

$$[T_A, T_B] = f_{AB}^C T_C \quad (2.13)$$

together with the metric tensor:

$$\langle T_A, T_B \rangle = g_{AB} \quad (2.14)$$

the connection defined by eq.(2.11) leads to the following connection coefficients:

$$\begin{aligned} \mathbb{L}_A T_B &= \Gamma_{AB}^C T_C \\ \Gamma_{AB}^C &= f_{AB}^C - g_{AD} g^{CE} f_{BE}^D - g_{BD} g^{CE} f_{AE}^D \end{aligned} \quad (2.15)$$

which are constant numbers.

Equivalently, we can define the Levi–Civita–Nomizu connection starting from the dual description of the solvable Lie algebra in terms of Maurer–Cartan equations. Let e^A be a basis of Maurer–Cartan forms dual to the generators T_B , namely $e^A(T_B) = \delta_B^A$. We have

$$de^C = \frac{1}{2} f_{AB}^C e^A \wedge e^B \quad (2.16)$$

Interpreting e^A as the vielbein over the solvable group manifold we write the vanishing torsion equation

$$0 = de^A + \omega^{AB} \wedge e^C g_{BC} \quad (2.17)$$

where $\omega^{AB} = -\omega^{BA}$ is the standard $\mathfrak{so}(n)$ –Lie algebra valued spin connection ($n = \dim(\mathcal{S})$). The relation between the two descriptions is immediate:

$$\omega^{AB} = \Gamma_{DE}^A g^{DB} e^E \quad (2.18)$$

where the tensor $\Gamma_{DE}^A g^{DB}$ is automatically antisymmetric in force of its definition.

Given the connection coefficients the differential geodesic equations can be immediately written. In the chosen basis the tangent vector to the geodesic is described by n fields $Y^A(t)$ which depend on the affine parameter t along the curve. The geodesic equation is given by the following first order differential system:

$$\frac{d}{dt} Y^A + \Gamma_{BC}^A Y^B Y^C = 0. \quad (2.19)$$

The above equation contains two data:

- 1) the structure constants of the solvable Lie algebra $f_{AB}{}^C$,
- 2) the metric tensor g_{AB} .

As we emphasized the second datum, namely the metric, comes down, in the Ptolemaic system from the geometric interpretation of the differential system (2.19) as geodesic equations on a Riemannian Einstein manifold and it is the metric of that manifold what eventually predetermines g_{AB} .

Let us now implement our Copernican Revolution and let us forget for a moment about the Riemannian structure. The system (2.19) is just a non linear differential system defined over the dual \mathcal{S}^* of a Solvable Lie algebra \mathcal{S} . How could we directly derive such a differential system and may be established its Liouville integrability from the very structure of \mathcal{S} ? Is there a way of deriving the metric tensor implicitly contained in (2.19) from the Lie algebra \mathcal{S} ? There is.

2.2 The Copernican Revolution: Poissonian structure of \mathcal{S} and Liouville integrability

Given the solvable Lie algebra \mathcal{S} we can consider the co-adjoint orbits of the corresponding solvable group $\exp[\mathcal{S}]$. In simple language this amounts to consider functions defined over the dual Lie algebra \mathcal{S}^* . An element of \mathcal{S}^* is a linear functional

$$\mathcal{L} \quad : \quad \mathcal{S} \rightarrow \mathbb{C} \text{ or } \mathbb{R} \quad (2.20)$$

and using a general theorem in linear algebra the dual of a finite dimensional vector space is isomorphic to the same space. In practice, given a basis T_A of \mathcal{S} we immediately obtain the dual basis \mathcal{L}^A by defining

$$\mathcal{L}^A(T_B) = \delta_B^A. \quad (2.21)$$

Any dual Lie algebra element $\mathfrak{Y} \in \mathcal{S}^*$ can be written as a linear combination of the dual generators

$$\mathfrak{Y} = Y_A \mathcal{L}^A \quad (2.22)$$

and the most general function ϕ on \mathcal{S}^* is actually a function of the n coordinates Y_A . Given any two such functions ϕ_1 and ϕ_2 we can define their Lie–Poisson bracket in the following manner:

$$\{\phi_1, \phi_2\} = f_{AB}{}^C Y_C \frac{\partial \phi_1}{\partial Y_A} \frac{\partial \phi_2}{\partial Y_B}. \quad (2.23)$$

In this way the space of co-adjoint orbits becomes a Poisson manifold, independently from the existence of any metric $<, >$ on \mathcal{S} . Then one is allowed to consider evolution equations of the following form:

$$\frac{d}{dt} Y_A + \{Y_A, \mathcal{H}\} = 0 \quad (2.24)$$

where $\mathcal{H} = \mathcal{H}(Y)$ is some function on the dual Lie algebra \mathcal{S}^* that we can regard as the Hamiltonian.

The question is whether the geodesic equations (2.19) can be put in the hamiltonian form (2.24), namely, whether there exists a hamiltonian function, necessarily quadratic, which reproduces the Levi-Civita–Nomizu connection. The answer is obviously yes. It suffices to write

$$\mathcal{H} \equiv \frac{1}{2} Y_A Y_B g^{AB} \quad (2.25)$$

and identify the variables Y^A and Y_B through the relation

$$Y_A = g_{AB} Y^B . \quad (2.26)$$

Both in eq. (2.25) and in eq.(2.26) there appears the metric g_{AB} defined by the non degenerate normal form $<, >$. Hence it may seem that we made no real progress and we simply rewrote the same equations in a different style. The notion of the metric tensor is still essential and it looks external to the pure Lie algebraic structure. It is not so, as it appears from the following argument.

2.2.1 Liouville integrability

The key point we would like to emphasize is that the definition of the Lie–Poisson bracket (2.23) depends only on the structure constants of the algebra \mathcal{S} and nothing else, so it is intrinsic to the algebra. Let us next recall the notion of Liouville integrability.

Definition 2.1 *A symplectic manifold of dimension $2m$ endowed with a Lie–Poisson bracket $\{, \}$ is Liouville integrable if there exists m functionally independent functions $\Phi_i(\mathcal{Y})$ of its $2m$ coordinates \mathcal{Y}_I that are in involution. Namely we must have:*

$$\forall i, j = 1, \dots, m \quad : \quad \{\Phi_i, \Phi_j\} = 0 \quad (2.27)$$

and

$$\text{rank} \left(\frac{\partial \Phi_i}{\partial \mathcal{Y}_J} \right) = m . \quad (2.28)$$

When these conditions are fulfilled any of the functions Φ_i can be chosen as the hamiltonian \mathcal{H} and the corresponding Euler equations

$$0 = \frac{d}{dt} \mathcal{Y}_I + \{\mathcal{Y}_I, \mathcal{H}\} \quad (2.29)$$

are completely integrable, since they admit m first integrals of the motion $\Phi_i(\mathcal{Y})$.

Let us now envisage the following:

2.2.2 Scenario

Given the solvable Lie algebra \mathcal{S} , whose dimension we denote

$$d_{\mathcal{S}} \equiv \dim \mathcal{S} , \quad (2.30)$$

imagine that with respect to its intrinsic Lie–Poissonian structure (2.23) there exist $p_{\mathcal{S}}$ functions $\mathfrak{h}_{\alpha}(Y)$ of the coordinates Y_A ($A = 1, \dots, d_{\mathcal{S}}$) on \mathcal{S}^* that are in involution

$$\{\mathfrak{h}_{\alpha}, \mathfrak{h}_{\beta}\} = 0 \quad (\alpha, \beta = 1, \dots, p_{\mathcal{S}}) \quad (2.31)$$

with the integer $p_{\mathcal{S}}$ lying in the range:

$$d_{\mathcal{S}} \geq p_{\mathcal{S}} \geq \left\lceil \frac{d_{\mathcal{S}}}{2} \right\rceil \quad (2.32)$$

and suppose furthermore that, having defined the integer

$$c \equiv 2p_{\mathcal{S}} - d_{\mathcal{S}} , \quad (2.33)$$

precisely c of the functions $\mathfrak{h}_{\alpha}(Y)$, which, for this reason we rename $\mathcal{C}_{\ell}(Y)$ ($\ell = 1, \dots, c$), are *Casimirs*, namely they commute with all the coordinates:

$$\{Y_A, \mathcal{C}_{\ell}\} = 0 \quad ; \quad (\ell = 1, \dots, c ; A = 1, \dots, d_{\mathcal{S}}) . \quad (2.34)$$

Under these hypotheses, by rearranging the set of involutive functions as it follows:

$$\{\mathfrak{h}_{\alpha}\} = \left\{ \underbrace{\Phi_i(Y)}_{i=1, \dots, m_{\mathcal{S}}} , \underbrace{\mathcal{C}_{\ell}(Y)}_{\ell=1, \dots, c} \right\} \quad (2.35)$$

we obtain the following situation. The *level surfaces* $\mathfrak{P}_{r_1, \dots, r_c}$ defined by setting the c Casimirs to fixed values

$$\mathfrak{y} \in \mathfrak{P}_{r_1, \dots, r_c} \Leftrightarrow \begin{cases} \mathcal{C}_1(\mathfrak{y}) & = & r_1 \in \mathbb{R} \text{ (or } \mathbb{C}) \\ \mathcal{C}_2(\mathfrak{y}) & = & r_2 \in \mathbb{R} \text{ (or } \mathbb{C}) \\ \dots & \dots & \dots \\ \mathcal{C}_c(\mathfrak{y}) & = & r_c \in \mathbb{R} \text{ (or } \mathbb{C}) \end{cases} \quad (2.36)$$

are by definition manifolds of even dimension:

$$\begin{aligned} \dim \mathfrak{P}_{r_1, \dots, r_c} &= 2m_{\mathcal{S}} \\ m_{\mathcal{S}} &= \frac{d_{\mathcal{S}} - c}{2} = d_{\mathcal{S}} - p_{\mathcal{S}} \end{aligned} \quad (2.37)$$

and on these manifolds there exist exactly $m_{\mathcal{S}}$ functions in involution, namely the $\Phi_i(Y)$ of eq.(2.35). It follows that each of these manifolds acquires a symplectic hamiltonian

structure. Naming \mathcal{Y}_I the $2m_S$ coordinates on $\mathfrak{P}_{r_1, \dots, r_c}$ and choosing as hamiltonian any linear combination of the pull-backs on $\mathfrak{P}_{r_1, \dots, r_c}$ of the functions $\Phi_i(Y)$

$$\mathcal{H}(\mathcal{Y}) = \text{pull-back on } \mathfrak{P}_{r_1, \dots, r_c} \text{ of } a^i \Phi_i(Y) \quad (2.38)$$

the corresponding Euler equation differential system

$$0 = \frac{d}{dt} \mathcal{Y}_I + \{\mathcal{Y}_I, \mathcal{H}\} \quad (2.39)$$

is by construction Liouville integrable. Indeed the pull-backs of the m_S functions $\Phi_i(Y)$ provide the necessary first integrals of motion.

2.2.3 Realization of this scenario on $\mathbb{B}(A_{N-1})$

The above described scenario is precisely realized in a very notable case, namely that of the Borel subalgebra $\mathbb{B}(A_{N-1}) \subset A_{N-1}$. As it is well known the simple Lie algebra A_{N-1} , identified by the following Dynkin diagram:

$$A_{N-1} \quad \begin{array}{ccccccc} \bigcirc & \text{---} & \bigcirc & \text{---} & \bigcirc & \cdots & \bigcirc & \text{---} & \bigcirc & \text{---} & \bigcirc \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_{N-3} & & \alpha_{N-2} & & \alpha_{N-1} \end{array}$$

is the abstract form of the Lie algebra $\mathfrak{sl}(N; \mathbb{C})$ of complex traceless matrices in dimension N . The corresponding Borel subalgebra $\mathbb{B}_N \equiv \mathbb{B}(A_{N-1})$ is simply given by the subset of all upper triangular traceless matrices. It is more convenient to relax the condition on the trace and consider the Borel subalgebra $\widehat{\mathbb{B}}_N \equiv \mathbb{B}(\mathfrak{gl}(N; \mathbb{C}))$ which is simply made by all upper triangular matrices. Reduction to \mathbb{B}_N will be performed by putting one of the Casimirs to the null value. Hence we define

$$\mathfrak{gl}(N; \mathbb{C}) \supset \widehat{\mathbb{B}}_N \ni \mathfrak{b} = \begin{pmatrix} \star & \star & \star & \star & \dots & \star \\ 0 & \star & \star & \star & \dots & \star \\ 0 & 0 & \star & \star & \dots & \star \\ 0 & 0 & 0 & \star & \dots & \star \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \star \end{pmatrix}. \quad (2.40)$$

The dimension of this solvable algebra is easily computed

$$\dim \widehat{\mathbb{B}}_N \equiv d_N = \frac{N(N+1)}{2}. \quad (2.41)$$

N	d_N	p_N	c_N	$2m$
2	3	2	1	2
3	6	4	2	4
4	10	6	2	8
5	15	9	3	12
6	21	12	3	18
7	28	16	4	24
8	36	20	4	32
9	45	25	5	40
10	55	30	5	50
11	66	36	6	60
12	78	42	6	72
13	91	49	7	84
14	105	56	7	98
15	120	64	8	112
16	136	72	8	128

Table 1: *In this table we give the dimensions of the Borel algebras $\mathbb{B}(\mathfrak{gl}(N))$, the corresponding number of functions in involutions, the corresponding number of Casimirs and the ensuing dimensions of the orbits for the first sixteen values of N .*

It was demonstrated by Arhangel'skii in [17] that this Poissonian manifold is Liouville integrable according to the scheme described in the previous subsection and the procedure to construct the required set of functions in involution was described in [17, 18]. Let us review these results. We distinguish the two cases where $N = 2\nu$ is even and where $N = 2\nu + 1$ is odd. The number p_N of functions in involution, the number of Casimirs and the ensuing even dimension of the orbits is displayed below:

	$d = \dim \widehat{\mathbb{B}}$	$p \equiv \# \text{ funct. in inv.}$	$c \equiv \# \text{ of Casim.}$	$2m = \dim \text{ of orbits}$
$\widehat{\mathbb{B}}_{2\nu}$	$\nu(2\nu + 1)$	$\nu^2 + \nu$	ν	$2\nu^2$
$\widehat{\mathbb{B}}_{2\nu+1}$	$(\nu + 1)(2\nu + 1)$	$\nu^2 + 2\nu + 1$	$\nu + 1$	$2(\nu^2 + \nu)$

(2.42)

For the reader's convenience in Table 1 we tabulated the first instances of such numbers.

2.2.4 Involutive hamiltonians

How are the functions $\mathfrak{h}_\alpha(Y)$ explicitly constructed?

First let us observe that a simple set of coordinates on the Borel subalgebra $\mathbb{B}(\mathfrak{gl}(N))$ is simply given by the entries of the upper triangular matrix \mathfrak{b} mentioned in eq.(2.40). Secondly, for reasons that will become clear later on, let us consider an $N \times N$ matrix L which satisfies the condition

$$(L \eta)^T = L \eta \quad (2.43)$$

where

$$\eta = \text{diag}(-1, +1, \dots, -1, +1, +1, \dots, +1) \quad , \quad p \leq q \quad ; \quad p + q = N \quad (2.44)$$

is any of the available choices of (p, q) signatures in dimension N . Irrespectively of the choice of η , the number of parameters contained in a matrix L satisfying (2.43) is always equal to $N(N+1)/2$ which is the dimension of the Borel algebra $\mathbb{B}(\mathfrak{gl}(N))$. Indeed there is a simple one-to-one map from the space of upper triangular matrices to the space of matrices satisfying eq.(2.43), which reads as follows:

$$\begin{aligned} \mathfrak{b} &\mapsto L = \mathfrak{b} + \eta \mathfrak{b}^T \eta \quad , \\ L &\mapsto \mathfrak{b} = L_{>} - \frac{1}{2} \text{diag} L \quad . \end{aligned} \quad (2.45)$$

In the above equation we have used the following convention. For any matrix M , we denote by $M_{>}$ its upper triangular part and by $M_{<}$ its lower triangular one including the diagonals.

Hence the space of matrices fulfilling eq.(2.43) provides a coordinate basis for the Borel algebra. Thanks to the simple relation (2.45), the use of different η tensors just amounts to a *linear coordinate transformation* on $\widehat{\mathbb{B}}_N$, in other words to a change of basis for the generators of the solvable Lie algebra. What is then the relevance of considering such different choices of η rather than focusing on a single conventional one? The answer is elementary. Each different η prepares a basis well adapted to the decomposition of the Lie algebra $\mathfrak{gl}(N)$ with respect to its subalgebra $\mathfrak{so}(p, q)$. Such a decomposition is necessary in order to study the geometry and the associated geodesic equations of the coset manifold

$$\mathcal{M}_{p,q} = \frac{\text{GL}(p+q)}{\text{SO}(p,q)} \quad . \quad (2.46)$$

All these (p, q) systems are integrable since all the manifolds $\mathcal{M}_{p,q}$ are metrically equivalent to the same solvable manifold $\exp[\widehat{\mathbb{B}}_{p+q}]$, equipped however with a different normal form $<, >_{p,q}$, which is positive definite only for $\{p=0, q=N\}$. The Liouville integrability of all these systems has the same common root, namely the existence of the p_N independent functions in involution that now we display and which is an intrinsic property of the Borel algebra $\widehat{\mathbb{B}}_{p+q}$.

The algorithm of constructing these functions, originally derived for $\{p = 0, q = N\}$ in [17] and [18], being generalized to the case under consideration, is the following.

Starting from the parameterization of $\widehat{\mathbb{B}}_N$ by means of the matrix L (2.45) fulfilling eq.(2.43) the complete set of p_N functions \mathfrak{h}_α that are involutive with respect to the Lie–Poisson bracket (2.23) is enumerated by an ordered pair of indices

$$\alpha = (a, b) \quad (2.47)$$

where:

$$\begin{aligned} a &= 0, \dots, \left\lfloor \frac{N}{2} \right\rfloor, \\ b &= 1, \dots, N - 2a. \end{aligned} \quad (2.48)$$

The functions \mathfrak{h}_{ab} can be iteratively derived from the following relation:

$$\begin{aligned} &\det \{(L - \lambda)_{ij} : a + 1 \leq i \leq N, 1 \leq j \leq N - a\} \\ &= \mathcal{E}_{a0} \left(\lambda^{N-2a} + \sum_{b=1}^{N-2a} \mathfrak{h}_{ab} \lambda^{N-2a-b} \right), \quad a = 0, \dots, \left\lfloor \frac{N}{2} \right\rfloor \end{aligned} \quad (2.49)$$

where, by definition, \mathcal{E}_{a0} is the coefficient of the power λ^{N-2a} .

As we know from the previous discussions c_N of these *generalized hamiltonians* are actually Casimirs.

For example, for the case $N = 3$ that we study in detail in the next sections the functions \mathfrak{h}_{02} , \mathfrak{h}_{03} are pure hamiltonians while \mathfrak{h}_{01} , \mathfrak{h}_{11} are Casimir functions.

2.3 Integrability and the metric $\langle, \rangle_{p,q}$

Having established in purely algebraic intrinsic terms the integrability of the differential system based on the solvable Lie algebra $\mathcal{S} = \widehat{\mathbb{B}}_N$, we can now answer the question about the origin of the metric form $\langle, \rangle_{p,q}$ which, in the Ptolemaic system, leads to the very same differential equations. The problem is very simply solved by recalling the quadratic form (2.25) of the hamiltonian which yields the geodesic equations associated with the Levi–Civita–Nomizu connection. The metric g_{AB} is defined by the coefficients appearing in general unique *quadratic* hamiltonian function $\mathcal{H} = \mathfrak{h}_{02}$. This view-point is actually very efficient. Changing the metric signature η amounts, as we have emphasized, to a change of basis on the solvable Lie algebra. Using all the time the same formula (2.49) to calculate the hamiltonians \mathfrak{h}_α , but varying the choice of η in the definition of L (2.45), results in a change of the coefficients in the quadratic hamiltonian \mathfrak{h}_{α_0} , namely in a change of metric g_{AB} over the Lie algebra. All the various coset models with indefinite signature are covered in this way by a unique algorithm.

2.4 Integrability and solvable subalgebras

In view of the above results on the integrability of all Borel algebras $\widehat{\mathbb{B}}_N$ there arises a natural question about the relation of this integrability with the possible integrability of its subalgebras. Indeed in view of the theorem that states that every linear representation of a solvable algebra \mathcal{S} admits a basis where all the elements $X \in \mathcal{S}$ are represented by an upper triangular matrix it follows that any solvable Lie algebra can be regarded as a subalgebra $\mathcal{S} \subset \widehat{\mathbb{B}}_N$ for a suitable choice of N .

Hence let us consider a generic subalgebra $\mathcal{S} \subset \widehat{\mathbb{B}}_N$ and name $\tilde{\mathfrak{h}}_\alpha$ the pull-back on \mathcal{S}^* of the p_N generalized hamiltonians defined on $\widehat{\mathbb{B}}_N$. In particular we are just interested in the differential system constructed on \mathcal{S}^* where the Hamiltonian $\tilde{\mathcal{H}}$ is the pull-back of the unique quadratic hamiltonian $\mathcal{H} = \mathfrak{h}_{02}$ constructed on $\widehat{\mathbb{B}}_N$. From the mathematical point of view there might be other choices but from the physical point of view this is the only relevant one. So let us put

$$\tilde{\mathcal{H}} = \tilde{\mathfrak{h}}_{02} \quad (2.50)$$

and let us divide the $\widehat{\mathbb{B}}_N$ coordinates in two subsets, those along \mathcal{S} and those normal to \mathcal{S}

$$\{Y_A\} = \left\{ \underbrace{X_i}_{\in \mathcal{S}}, \underbrace{W_\alpha}_{\notin \mathcal{S}} \right\}. \quad (2.51)$$

Next let us compare the evolution equations of the subalgebra \mathcal{S} with the pull-back on \mathcal{S} of the \mathbb{B}_N equations. In view of our choice (2.50) we have

$$\tilde{\mathcal{H}}(X) = \mathcal{H}(X, W)|_{W=0} \equiv \mathcal{H}(X, 0). \quad (2.52)$$

The \mathcal{S} -equations are simply read

$$\begin{aligned} \frac{d}{dt} X_i &= -\{X_i, \tilde{\mathcal{H}}\} \\ &= -f_{ij}^{\quad k} X_k \partial^j \tilde{\mathcal{H}}. \end{aligned} \quad (2.53)$$

while the pull-back on \mathcal{S} of the $\widehat{\mathbb{B}}_N$ equations is

$$\begin{aligned} \frac{d}{dt} X_i &= -\{X_i, \mathcal{H}\}|_{W=0} \\ &= -f_{ij}^{\quad k} X_k \partial^j \tilde{\mathcal{H}} - f_{i\alpha}^{\quad k} X_k \partial^\alpha \mathcal{H}|_{W=0}, \end{aligned} \quad (2.54)$$

$$\begin{aligned} \frac{d}{dt} W_\alpha &= -\{W_\alpha, \mathcal{H}\}|_{W=0} \\ &= -f_{\alpha\beta}^{\quad k} X_k \partial^\beta \mathcal{H}|_{W=0} - f_{\alpha i}^{\quad k} X_k \partial^i \tilde{\mathcal{H}}. \end{aligned} \quad (2.55)$$

The conditions necessary for the system (2.54,2.55) to reduce consistently to the system (2.53) are the following ones:

$$\partial^\beta \mathcal{H}|_{W=0} = 0, \quad (2.56)$$

$$f_{\alpha i}^{\quad k} = 0. \quad (2.57)$$

The first condition is satisfied if there are no mixed coefficients in the metric implicitly defined by the hamiltonian, namely if

$$g^{i\alpha} = 0 . \quad (2.58)$$

The second condition implies that the decomposition of the Borel algebra $\widehat{\mathbb{B}}_N$ with respect to its solvable subalgebra \mathcal{S} should be reductive. Altogether the conditions for the consistent reduction of the system (2.54,2.55) to the system (2.53) can be written as follows:

$$\widehat{\mathbb{B}}_N = \mathcal{S} \oplus \mathcal{S}_\perp , \quad (2.59)$$

$$\langle \mathcal{S}, \mathcal{S}_\perp \rangle = 0 , \quad (2.60)$$

$$[\mathcal{S}, \mathcal{S}] \subset \mathcal{S} , \quad (2.61)$$

$$[\mathcal{S}, \mathcal{S}_\perp] \subset \mathcal{S}_\perp . \quad (2.62)$$

Under the hypotheses (2.59–2.62) a geodesic on the manifold $(\exp[S], \widetilde{\langle, \rangle})$ satisfies the geodesic equations on the embedding manifold $(\exp[\widehat{\mathbb{B}}_N], \langle, \rangle)$. The latter is a symmetric space, namely $GL(N, \mathbb{R})/SO(p, N - p)$, so that $(\exp[S], \widetilde{\langle, \rangle})$ is revealed to be a *geodesically complete submanifold* of a symmetric space. In force of a general theorem [34] this implies that also $(\exp[S], \widetilde{\langle, \rangle})$ is a *symmetric space*.

Hence the exceptional cases of homogeneous solvable manifolds $(\exp[\mathcal{S}], \widetilde{\langle, \rangle})$ that are not symmetric spaces (compare with the classification of homogeneous special geometries, [30]) are based on solvable Lie algebras \mathcal{S} whose embedding in $\widehat{\mathbb{B}}_N$ is not reductive and violates eq.s (2.59–2.62). In order to establish their Liouville integrability one has to study the pull-back on \mathcal{S} of the p_N involutive hamiltonians \widetilde{h}_α (see Table 1) and ascertain how many of them remain in involution among themselves and with the unique hamiltonian $\widetilde{\mathcal{H}} = \widetilde{h}_{02}$. Such a study involves several complicacies and it is postponed to a subsequent publication. For the case of symmetric spaces, instead, the proper reduction of the hamiltonians and the consequent integrability is guaranteed a priori from the proper reduction of the Lax representation that we discuss in the next section.

3 Triangular embedding in $SL(N; \mathbb{R})/SO(p, N - p; \mathbb{R})$ and integrability of the Lorentzian cosets U/H^*

As already recalled in the introduction it was realized in recent years that the field equations of supergravity describing either

- 1) space-like p -branes as the cosmic billiards, or
- 2) time-like p -branes as several rotational invariant black-holes in $D = 4$ and more general solitonic branes in diverse dimensions

reduce to geodesic equations on coset manifolds of the type

$$\mathcal{M} = \frac{\mathbb{U}}{\mathbb{H}} \quad \text{or} \quad \mathcal{M}^* = \frac{\mathbb{U}}{\mathbb{H}^*} \quad (3.1)$$

where $\mathbb{U} = \exp[\mathbb{U}]$ is the group manifold generated by the duality algebra \mathbb{U} relevant to the considered supergravity model in the considered dimensions. As a rule without exceptions \mathbb{U} is always some *non-compact real form* of a complex Lie algebra $\mathbb{U}_{\mathbb{C}}$. The Lie algebra \mathbb{H} of the subgroup \mathbb{H} is instead the unique maximal compact subalgebra of \mathbb{U} . The coset \mathcal{M} corresponds to the case of space-like p -branes, in particular cosmic billiards. In the second coset \mathcal{M}^* , pertaining to the case of time-like branes, the Lie algebra \mathbb{H}^* of \mathbb{H}^* is another non-compact real section of the complexification $\mathbb{H}_{\mathbb{C}}$, while \mathbb{U} remains the same. It happens indeed the following situation always occurs: the real duality algebra \mathbb{U} admits as proper subalgebras a few different instances of real sections of $\mathbb{H}_{\mathbb{C}}$ which necessarily include the maximally compact one \mathbb{H} .

We recall few illustrative examples. Consider the case of maximal supersymmetry with 32 supercharges. The duality algebra in $D = 4$ is $\mathbb{U} = E_{7(7)}$ whose maximal compact subalgebra is $\mathbb{H} = \mathfrak{su}(8) \subset E_{7(7)}$. The complexification of this is $\mathbb{H}_{\mathbb{C}} = \mathfrak{sl}(8, \mathbb{C})$. Aside from $\mathfrak{su}(8)$ the other real sections of $\mathfrak{sl}(8, \mathbb{C})$ which are subalgebras of $E_{7(7)}$ are the following ones: $\mathbb{H}^* = \mathfrak{sl}(8, \mathbb{R}) \subset E_{7(7)}$ and $\mathbb{H}^* = \mathfrak{su}^*(8) \subset E_{7(7)}$. For the same number of supercharges the duality algebra in $D = 5$ is $\mathbb{U} = E_{6(6)}$. Here we have $\mathbb{H} = \mathfrak{usp}(8)$, $\mathbb{H}_{\mathbb{C}} = \mathfrak{sp}(8, \mathbb{C})$, $\mathbb{H}^* = \mathfrak{usp}(4, 4)$ or $\mathbb{H}^* = \mathfrak{sp}(8, \mathbb{R})$.

Consider instead the case of 1/2 maximal supersymmetry, namely supergravity with 16 supercharges. In $D = 3$ the theory with $6 + n$ supervector multiplets has the following duality algebra $\mathbb{U} = \mathfrak{so}(8, 8 + n)$. The corresponding maximal compact subalgebra is $\mathbb{H} = \mathfrak{so}(8) \times \mathfrak{so}(8 + n)$ and the available \mathbb{H}^* are: $\mathbb{H}^* = \mathfrak{so}(8 - p, p) \times \mathfrak{so}(8 + n - p, p)$ ($p=1, 2, 3, 4$).

For all these cases we can make the following statement.

Statement 3.1 << Let N be the real dimension of the fundamental representation of \mathbb{U} . For each choice of \mathbb{H} or \mathbb{H}^* there exist a suitable integer $p \leq \lfloor \frac{N}{2} \rfloor$ and a diagonal metric

$$\eta = \text{diag}(\underbrace{-1, +1, \dots, -1, +1}_{2p}, \underbrace{+1, +1, \dots, +1}_{N-2p}) , \quad (3.2)$$

such that we have a canonical embedding

$$\begin{aligned} \mathbb{U} &\hookrightarrow \mathfrak{sl}(N; \mathbb{R}) , \\ \mathbb{U} \supset \mathbb{H}^* &\hookrightarrow \mathfrak{so}(p, N - p; \mathbb{R}) \subset \mathfrak{sl}(N; \mathbb{R}) . \end{aligned} \quad (3.3)$$

This embedding is determined by the choice of the basis where $\text{Solv}(\mathbb{U}/\mathbb{H}^*)$ is made by upper triangular matrices. In the same basis the elements of \mathbb{K} are η -symmetric matrices while those of \mathbb{H}^* are η -antisymmetric ones, namely:

$$\begin{aligned} \forall K \in \mathbb{K} : \quad & \eta K^T = K^T \eta , \\ \forall H \in \mathbb{H}^* : \quad & \eta H^T = -H^T \eta . \end{aligned} \quad (3.4)$$

>>

Just as in the case of cosmic billiards the embedding of the Riemannian coset U/H into the universal covering coset $SL(N; \mathbb{R})/SO(N; \mathbb{R})$ provided the key to obtain an explicit integration algorithm for the associated first order geodesic equations, in the same way the embedding (3.3) of the pseudo-Riemannian U/H^* into $SL(N; \mathbb{R})/SO(p, N-p; \mathbb{R})$ provides the key to extend the same integration algorithm also to this indefinite metric symmetric spaces. Indeed that algorithm is defined for $SL(N; \mathbb{R})/SO(p, N-p; \mathbb{R})$ and it has the property that if initial data are defined in a submanifold U/H^* where $U \subset SL(N; \mathbb{R})$ and $H^* \subset SO(p, N-p; \mathbb{R})$, then the entire *time flow* occurs in the same submanifold. This property is a simple consequence of the following argument. Let $L \in \mathbb{K}$ be an η -symmetric matrix satisfying property (2.43) which lies in the subspace \mathbb{K} corresponding to the symmetric decomposition $\mathbb{U} = \mathbb{H}^* \oplus \mathbb{K}$ of the Lie subalgebra $\mathbb{U} \subset \mathfrak{gl}(N; \mathbb{R})$. By construction the corresponding connection $W \equiv L_{>} - L_{<}$ lies in \mathbb{H}^* , namely we have $W \in \mathbb{H}^*$. It follows that $[W, L] \in \mathbb{K}$. Hence if $L(0) \in \mathbb{K}$, the evolution equation $\dot{L} + [W, L] = 0$ never brings $L(t)$ out of that space.

Hence the embedding (3.3) suffices to define explicit integration formulae for all supergravity time-like p -branes based on pseudo-Riemannian symmetric spaces.

Let us review the steps of the procedure.

1. First one defines a coset representative for U/H^* just as in the Riemannian case, namely:

$$\mathbb{L}(\phi) = \prod_{I=m}^{I=1} \exp[\varphi_I E^{\alpha_I}] \exp[h_i \mathcal{H}^i] \quad (3.5)$$

where the roots pertaining to the solvable Lie algebra are ordered in ascending order of height ($\alpha_I \leq \alpha_J$ if $I < J$), \mathcal{H}^i denote the non compact Cartan generators and the product of matrix exponentials appearing in (3.5) goes from the highest on the left, to lowest root on the right. In this way the parameters $\{\phi\} \equiv \{\varphi_I, h_i\}$ have a precise and uniquely defined correspondence with the fields of supergravity by means of dimensional oxidation [5, 6]. Thanks to the mapping (2.45), the same upper triangular matrix $\mathbb{L}(\phi)$ can be regarded as a coset representative for U/H or for U/H^* .

2. Restricting all the fields ϕ of supergravity to pure *time dependence*⁶ $\phi = \phi(t)$, the coset representative becomes also a function of *time* $\mathbb{L}(\phi(t)) = \mathbb{L}(t)$ and we define the Lax operator $L(t)$ and the connection $W(t)$ as follows:

$$\begin{aligned} L(t) &= \sum_i \text{Tr} \left(\mathbb{L}^{-1} \frac{d}{dt} \mathbb{L} K_i \right) K_i, \\ W(t) &= \sum_\ell \text{Tr} \left(\mathbb{L}^{-1} \frac{d}{dt} \mathbb{L} H_\ell \right) H_\ell \end{aligned} \quad (3.6)$$

⁶In the case of time like p -branes such as black-holes *time* is actually the radial coordinate.

where K_i and H_ℓ denote an orthonormal basis of generators for \mathbb{K} and \mathbb{H}^* , respectively

$$\begin{aligned}
\text{Tr}(K_i, K_j) &= \text{diag} \left(\underbrace{+, \dots, +}_{n-r}, \underbrace{-, \dots, -}_r \right), \\
\text{Tr}(H_i, H_j) &= 2 \text{diag} \left(\underbrace{+, \dots, +}_r, \underbrace{-, \dots, -}_{m-r} \right), \\
\text{Tr}(K_i, H_j) &= 0, \\
n &\equiv \dim \left(\frac{\mathbb{U}}{\mathbb{H}^*} \right) = \dim \left(\frac{\mathbb{U}}{\mathbb{H}} \right), \\
m &\equiv \dim(\mathbb{H}^*) = \dim(\mathbb{H}), \\
r &\equiv \# \text{ compact generators of } \mathbb{K} = \# \text{ non-compact generators of } \mathbb{H}.
\end{aligned} \tag{3.7}$$

3. With these definitions the field equations of the supergravity time-like p -brane or black-hole, which are just the geodesic equations for the manifold \mathbb{U}/\mathbb{H}^* in the solvable parametrization, reduce to the single matrix valued Lax equation [9]

$$\frac{d}{dt} L(t) + [W(t), L(t)] = 0 \tag{3.8}$$

which is the compatibility condition for the linear system exhibiting the iso-spectral property of the Lax operator L

$$L \Psi = \Psi \Lambda, \tag{3.9}$$

$$\frac{d}{dt} \Psi = W \Psi \tag{3.10}$$

where

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N) \tag{3.11}$$

is the diagonal matrix of eigenvalues and $\Psi(t)$ is the eigenmatrix.

4. If we are able to write the general integral of the Lax equation, depending on $n = \dim(\mathbb{U}/\mathbb{H}^*)$ integration constants, then comparison of the definition of the Lax operator (3.6,3.5) with its explicit form in the integration reduces the differential equations of supergravity to quadratures

$$\frac{d}{dt} \phi(t) = F(t) = \text{known function of time.} \tag{3.12}$$

3.1 The integration algorithm for the Lax Equation

Let us assume that we have explicitly constructed the embedding (3.3). In this case, in the decomposition

$$\mathbb{U} = \mathbb{K} \oplus \mathbb{H}^* \quad (3.13)$$

of the relevant Lie algebra \mathbb{U} , the matrices representing the elements of \mathbb{K} are all η -symmetric while those representing the elements of \mathbb{H}^* are all η -antisymmetric as we have already pointed out. Furthermore the matrices representing the solvable Lie algebra $\text{Solv}(\mathbb{U}/\mathbb{H}^*)$ are all upper triangular. These are the necessary and sufficient conditions to apply to the relevant Lax equation (3.8) the integration algorithm originally described in [19] and reviewed in [9, 10]. The key point is that the connection $W(t)$ appearing in eq.(3.8) is related to the Lax operator by means of the already recalled algebraic projection operator as follows:

$$W = \Pi(L) := L_{>} - L_{<} . \quad (3.14)$$

The relation (3.14) is nothing else but the statement that the coset representative $\mathbb{L}(\phi)$ ($= L + W$) from which the Lax operator is extracted is taken in the solvable parametrization.

The only relevant new feature distinguishing the indefinite metric case from the definite one is the discussion of the spectral types.

3.1.1 Spectral types

From eqs.(2.43) it follows that, generically, the Lax operator $L(0)$ is not a symmetric matrix. It is such only in the Euclidean case ($p = 0$ and $q = N$). Therefore $L(0)$ eigenvalues are generically complex numbers. We will concentrate on the case when $L(0)$ is a simple matrix, i.e. all its eigenvalues are distinct. Then, in order for $L(0)$ to be real, its eigenvalues have to group in k complex conjugated pairs and $N - 2k$ real eigenvalues, where k is some fixed integer in the range $0 \leq k \leq p$. Obviously, Lax matrices corresponding to different values of k can not be related by a similarity transformation. Thus the integer k parameterizes inequivalent initial data $L_k(0)$ that we decide to name *spectral types*. For the metric (2.44) one can choose a basis in the space of eigenvalues where their complex conjugation properties become

$$\begin{aligned} \lambda_{2\alpha-1}^* &= \lambda_{2\alpha} , & \alpha = 1, \dots, k , & \quad 0 \leq k \leq p , \\ \lambda_{\alpha}^* &= \lambda_{\alpha} , & \alpha = 2k + 1, \dots, N . \end{aligned} \quad (3.15)$$

Simple inspection of eqs. (3.15) shows that the corresponding matrix eigenvalues Λ_k (3.11) and its complex conjugated matrix $\bar{\Lambda}_k$ are actually related by a similarity transformation

$$\bar{\Lambda}_k = T_k \Lambda_k T_k^{-1} \quad (3.16)$$

which will be useful in what follows. Here, the complex symmetric $N \times N$ -matrix T_k has a very simple block-diagonal structure

$$T_k := \text{diag} (B, \dots, B, -\mathbf{1}_{N-2k}) \quad (3.17)$$

and satisfies the properties

$$\bar{T}_k T_k = \mathbf{1}_N , \quad T_k \eta T_k = \eta . \quad (3.18)$$

The matrix T_k comprises k sub-blocks given by the (2×2) -matrix B , defined below:

$$B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} , \quad B^2 = -\mathbf{1}_2 , \quad (3.19)$$

and the bottom $((N - 2k) \times (N - 2k))$ -matrix sub-block which is proportional to the $((N - 2k) \times (N - 2k))$ -unity matrix $\mathbf{1}_{N-2k}$; i is the imaginary unity. The matrices Λ_k ($k = 0, 1, \dots, p$) satisfying eq. (3.16) form the initial data for the $L(0)$ eigenvalues.

Now, let us turn to constructing the initial data for the $L(0)$ -eigenmatrix $\Psi(0)$ (3.9). If the Lax eigenvalues are complex (i.e., $\bar{\Lambda}_k \neq \Lambda_k$), then the eigenmatrix $\Psi(0)$ has to be complex as well in order for the Lax matrix

$$L_k(0) = \Psi_k(0) \Lambda_k \Psi_k^{-1}(0) \quad (3.20)$$

to be real

$$\bar{\Psi}_k(0) \bar{\Lambda}_k (\bar{\Psi}_k(0))^{-1} = \Psi_k(0) \Lambda_k \Psi_k^{-1}(0) . \quad (3.21)$$

Besides the constraint (3.21), the complex eigenmatrix $\Psi(0)$ which diagonalizes the initial Lax operator $L(0)$ should satisfy one more constraint [19]

$$\Psi_k^T(0) \eta \Psi_k(0) = \eta , \quad (3.22)$$

namely it should belong to the complexified group $\text{SO}(p, N - p; \mathbb{C})$. Let us introduce the new set of block-diagonal complex, symmetric $N \times N$ -matrices

$$\hat{T}_k := \text{diag} \left(\frac{1}{\sqrt{2}} (\mathbf{1}_2 - B), \dots, \frac{1}{\sqrt{2}} (\mathbf{1}_2 - B), \mathbf{1}_{N-2k} \right) \quad (3.23)$$

that satisfy the following relations among themselves, with the metric η (2.44) and with the previously introduced set T_k (3.17):

$$\bar{\hat{T}}_k \hat{T}_k = \mathbf{1}_N , \quad \hat{T}_k \eta \hat{T}_k = \eta , \quad \bar{\hat{T}}_k T_k = -\hat{T}_k . \quad (3.24)$$

Using eqs.(3.16) and (3.24) by straightforward calculation we can verify that the complex eigenmatrix

$$\Psi_k(0) := \mathcal{O}_0 \hat{T}_k \quad (3.25)$$

satisfies both constraints (3.21) and (3.22) if the introduced *real* matrix \mathcal{O}_0 , entering into eq. (3.25), satisfies the pseudo-orthogonality condition

$$\mathcal{O}_0^T \eta \mathcal{O}_0 = \eta , \quad (3.26)$$

namely if $\mathcal{O}_0 \in \text{SO}(p, N - p; \mathbb{R})$.

We conclude that the initial data for the Lax operator are represented in the following way:

$$\begin{aligned} L_k(0) &= \mathcal{O}_0 \widehat{T}_k \Lambda_k \widehat{T}_k^{-1} \mathcal{O}_0^{-1} \quad \text{where} \\ \mathcal{O}_0 &\in \text{SO}(p, N - p; \mathbb{R}) . \end{aligned} \quad (3.27)$$

Hence the manifold of solutions of the Lax equation for the case $\text{GL}(N, \mathbb{R})/\text{SO}(p, N - p; \mathbb{R})$, splits into $(p + 1)$ disconnected branches corresponding to the spectral types $k = 0, 1, \dots, p$.

When we consider a subsystem U/H^\star , having fixed p , the actual number of spectral types is determined by considering which normal forms

$$\widehat{\Lambda}_k \equiv \widehat{T}_k \Lambda_k \widehat{T}_k^{-1} \quad (3.28)$$

actually belong to the space \mathbb{K} . Indeed the spectral type k will be included in the integration algorithm if and only if

$$\widehat{\Lambda}_k \in \mathbb{K} . \quad (3.29)$$

The above discussion of spectral types actually coincides with the discussion of normal forms reported in [16]⁷. In the same paper the normal form $\widehat{\Lambda}_k$ was group-theoretically interpreted as an element of the subspace \mathbb{K} belonging to a subvector space of the form:

$$\widehat{\Lambda}_k \in \left(\frac{\mathfrak{sl}(2; \mathbb{R})}{\mathfrak{so}(1, 1; \mathbb{R})} \right)^k \times \mathfrak{so}(1, 1; \mathbb{R})^{N-k} \subset \mathbb{K} . \quad (3.30)$$

Furthermore in [16] the number of actually available normal forms (spectral types in our nomenclature) was shown to admit the following group theoretical interpretation:

$$\# \text{ of spectral types} = \text{rank} \left(\frac{H^\star}{H_c} \right) + 1 \quad (3.31)$$

where $H_c \subset H^\star$ is the maximal compact subgroup of H^\star .

3.1.2 The Kodama integration algorithm for $\frac{\text{SL}(N; \mathbb{R})}{\text{SO}(p, N - p; \mathbb{R})}$ revisited

Having clarified the fundamental issue of spectral types let us describe in full detail the adaptation of the Kodama integration algorithm [19] to the indefinite metric case.

Summarizing all our previous discussions the starting point of the algorithm is provided by the initial data listed below:

- a) The spectral type, codified by the choice of one of the $p + 1$ matrices \widehat{T}_k ($k = 0, 1, \dots, p$) (3.23). At fixed k we set:

$$\mathcal{T} := \widehat{T}_k . \quad (3.32)$$

⁷We would like to thank our friend and long term collaborator M. Trigiante who attracted our attention to the constructions of paper [16] when the results of the present paper were already in final form.

b) The eigenvalue vector:

$$\vec{\lambda} = \left\{ \lambda_1, \lambda_2, \dots, \lambda_{N-1}, -\sum_{i=1}^{N-1} \lambda_i \right\} . \quad (3.33)$$

If the spectral type is $k = 0$ all the eigenvalues are real. If the spectral type is $k = r$ then the first $2r$ eigenvalues are arranged in complex conjugate pairs such that $\lambda_{2i} = \bar{\lambda}_{2i-1}$ while the remaining $N - 2r$ are real.

c) The choice of an arbitrary element of the pseudo-rotational group $\mathcal{O}_0 \in \text{SO}(p, N - p; \mathbb{R})$, namely a real $N \times N$ matrix satisfying:

$$\mathcal{O}_0^T \eta \mathcal{O}_0 = \eta . \quad (3.34)$$

In terms of these initial data we define the following complex matrices:

$$\Psi(0) = \mathcal{O}_0 \mathcal{T} \in \text{SO}(p, N - p; \mathbb{C}) . \quad (3.35)$$

Next we construct a *time-dependent* $N \times N$ matrix $c(t)$ whose elements are defined by the following formula:

$$c_{ij}(t) = \sum_{k=1}^N \Psi_{ik}(0) \exp[-2 \text{Re}(\lambda_k) t] (\cos[-2 \text{Im}(\lambda_k) t] + i \sin[-2 \text{Im}(\lambda_k) t]) (\Psi^{-1}(0))_{kj} , \quad (3.36)$$

and we introduce the N -functions $D_\ell(t)$ ($D_0(t) := 0$) constructed as the determinants of the principal diagonal $\ell \times \ell$ sub-matrices of $c(t)$, namely:

$$D_\ell(t) = \text{Det} \left[\left(c_{ij}(t) \right)_{1 \leq i, j \leq \ell} \right] . \quad (3.37)$$

Using these building blocks we construct a *time-dependent*, real, pseudo-rotation matrix $\mathcal{O}(t)$ in the following way. First define a *time-dependent* complex matrix $\Psi(t)$ whose entries are given by

$$\Psi_{ij}(t) = \frac{\exp[-\text{Re}\lambda_j t] (\cos[-\text{Im}\lambda_j t] + i \sin[-\text{Im}\lambda_j t])}{\sqrt{D_i(t) D_{i-1}(t)}} \text{Det} \begin{pmatrix} c_{11} & \dots & c_{1,i-1} & \Psi_{1j}(0) \\ \vdots & \ddots & \vdots & \vdots \\ c_{i1} & \dots & c_{i,i-1} & \Psi_{ij}(0) \end{pmatrix} \quad (3.38)$$

and then we obtain the desired $\mathcal{O}(t)$ by setting

$$\mathcal{O}(t) \equiv \Psi(t) \mathcal{T}^{-1} . \quad (3.39)$$

It is a matter of direct verification that defined as above, the evolving $\text{SO}(p, N - p)$ group element $\mathcal{O}(t)$ is indeed real at all instants of *time*

$$\overline{\mathcal{O}}(t) = \mathcal{O}(t) , \quad (3.40)$$

thus $\mathcal{O}(t) \in \text{SO}(p, N - p; \mathbb{R})$. Finally the explicit form of the Lax operator solving Lax equation with the chosen set of initial conditions is given by

$$L(t) = \mathcal{O}(t) \hat{\Lambda} \mathcal{O}^{-1}(t) , \quad (3.41)$$

$$\hat{\Lambda} = \mathcal{T} \Lambda \mathcal{T}^{-1} \quad (3.42)$$

where:

$$\Lambda = \left(\begin{array}{c|c|c|c|c} \lambda_1 & 0 & \dots & 0 & 0 \\ \hline 0 & \lambda_2 & \dots & \dots & 0 \\ \hline \dots & \dots & \dots & \dots & \dots \\ \hline 0 & 0 & \dots & \lambda_{N-1} & 0 \\ \hline 0 & 0 & \dots & 0 & -\sum_{i=1}^{N-1} \lambda_i \end{array} \right) . \quad (3.43)$$

At first sight the reader might consider baroque the substitution

$$\exp[-2\lambda_j t] \mapsto \exp[-2\text{Re}(\lambda_k) t] (\cos[-2\text{Im}(\lambda_k) t] + i \sin[-2\text{Im}(\lambda_k) t]) \quad (3.44)$$

used both in eq. (3.36) and eq. (3.38). Actually such a substitution is very handy in order to verify the reality of the solution and also in order to understand its analytic structure. First of all it is fairly simple to check that the matrix $c_{ij}(t)$ and hence all of its minors are real. Secondly thanks to the same token one verifies that also $\mathcal{O}(t)$ is real and all of its entries are rational functions of exponentials, cosines and sines or square-roots thereof. The periodic trigonometric functions are absent for the spectral type $k = 0$ and appear only for $k \geq 1$. The appearance of these sines and cosines is the truly new feature due to the pseudo-Riemannian structure of the denominator group H^* .

For all spectral types the evolution of the Lax operator is given by a time dependent $\text{SO}(p, N - p; \mathbb{R})$ similarity transformation, starting from a normal form $\hat{\Lambda}$, but this normal form is diagonal only in the case of the spectral type $k = 0$ when all eigenvalues are real. In the other spectral types the real normal form $\hat{\Lambda}$ is non-diagonal. It has the structure discussed in the previous sections.

4 The paradigmatic example: $\text{SL}(3; \mathbb{R})/\text{SO}(1, 2; \mathbb{R})$ versus $\text{SL}(3; \mathbb{R})/\text{SO}(3; \mathbb{R})$

In the present section we illustrate the previously presented theory with a simple, yet paradigmatic example, that of the cosets

$$\mathcal{M}_{p,3} = \frac{\text{SL}(3; \mathbb{R})}{\text{SO}(p, 3 - p; \mathbb{R})} \quad (4.1)$$

where $p = 1$ is the new case with respect to the well known example of $p = 0$ (see for instance [10]). From the physical point of view the coset $\mathcal{M}_{0,3}$ provides the description of $D = 5$ pure gravity reduced three dimensions. Correspondingly the coset $\mathcal{M}_{1,3}$ is just related with time-like brane solutions of $D = 5$ pure gravity. It can also be related to several other interesting brane constructions, yet the viewpoint adopted in the present paper is mathematically oriented. We just want to illustrate by means of this examples the key points of the explained general constructions. In particular we aim at illustrating the relation between the choice of spectral type, the assignment of values to the generalized hamiltonians in involution and the analytic structure of the constructed integrals.

Hence discarding the well known case $p = 0$ for which we refer the reader to [10], we concentrate on the novel features of the case $p = 1$.

4.1 $\mathfrak{so}(1, 2; \mathbb{R})$ decomposition of the $\mathrm{SL}(3; \mathbb{R})$ Lie algebra

The starting point is to consider a basis of $\mathrm{SL}(3; \mathbb{R})$ generators well adapted to the Minkowskian coset $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(1, 2; \mathbb{R})$ rather than to the Euclidian coset $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3; \mathbb{R})$. This is easily constructed with the following procedure. First one multiplies all generators adapted to the euclidian basis by the Lorentzian metric tensor

$$\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (4.2)$$

then one redefines the diagonal Cartan generators by subtracting their trace and making them in this way traceless. The result is a set of 8 generators of the $\mathfrak{sl}(3; \mathbb{R})$ Lie algebra with the property that the last three close the Lie algebra of $\mathfrak{so}(1, 2; \mathbb{R})$, while the first five span a basis of the spin $s = 2$ representation of the same. Namely we have:

$$\begin{aligned} \mathcal{T}_A &= \left\{ \underbrace{K_i}_{i=1, \dots, 5}, \underbrace{J_a}_{a=1, \dots, 3} \right\}, \\ 0 &= \eta J_a^T + J_a \eta, \\ [J_a, J_b] &= \varepsilon_{abc} \eta^{cd} J_d, \\ [J_a, K_i] &= R(J_a)_{ij} K_j \end{aligned} \quad (4.3)$$

where ε_{abc} is the standard Levi-Civita antisymmetric symbol and $R(J_a)_{ij}$ denote the 5×5 matrices representing the $\mathfrak{so}(1, 2; \mathbb{R})$ generators in the spin $s = 2$ case.

Explicitly we have

$$\begin{aligned}
K_1 &= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad K_2 = \begin{pmatrix} -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix} , \\
K_3 &= \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad K_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} , \\
K_5 &= \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} ,
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
J_1 &= \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad J_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} , \\
J_3 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} .
\end{aligned} \tag{4.5}$$

As one can easily note, among the coset generators three are non-compact, while two are compact.

With reference to eq.(3.7) this means that in this case we have

$$r = 2 . \tag{4.6}$$

Explicitly the non-compact coset generators are the two Cartans $K_{1,2}$ and the off-diagonal generator K_4 . The two compact coset generators are instead $K_{3,5}$. Similarly the three subalgebra generators are distributed into the two non-compact ones $J_{1,3}$ and the compact one J_2 . The explicit form of the 5×5 matrices representing the generators J_i is the

following one:

$$\begin{aligned}
R(J_1) &= \begin{pmatrix} 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} ; \quad R(J_2) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} , \\
R(J_3) &= \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & \sqrt{3} & 0 & 0 & 0 \end{pmatrix} ,
\end{aligned} \tag{4.7}$$

and they are such that

$$[J_a, K_i] = R(J_a)_{ij} K_j . \tag{4.8}$$

4.2 The Lorentzian Lax operator

The next step is the definition of the Lax operator. According to our established conventions we define it as follows:

$$\begin{aligned}
L(t) &= Y_1(t) K_1 + Y_2(t) K_2 + \frac{1}{\sqrt{2}} \sum_{i=3}^5 Y_i(t) K_i \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} Y_1(t) - \frac{1}{\sqrt{6}} Y_2(t) & -\frac{1}{2} Y_3(t) & -\frac{1}{2} Y_5(t) \\ \frac{1}{2} Y_3(t) & -\frac{1}{\sqrt{2}} Y_1(t) - \frac{1}{\sqrt{6}} Y_2(t) & -\frac{1}{2} Y_4(t) \\ \frac{1}{2} Y_5(t) & -\frac{1}{2} Y_4(t) & \sqrt{\frac{2}{3}} Y_2(t) \end{pmatrix} .
\end{aligned} \tag{4.9}$$

The $\mathfrak{so}(1, 2; \mathbb{R})$ Lie-algebra-valued connection is then obtained from the Lax operator with the standard R -matrix rule

$$\begin{aligned}
W(t) &\equiv L_{>}(t) - L_{<}(t) \\
&= \begin{pmatrix} 0 & -\frac{1}{2} Y_3(t) & -\frac{1}{2} Y_5(t) \\ -\frac{1}{2} Y_3(t) & 0 & -\frac{1}{2} Y_4(t) \\ -\frac{1}{2} Y_5(t) & \frac{1}{2} Y_4(t) & 0 \end{pmatrix}
\end{aligned} \tag{4.10}$$

and duely satisfies the condition

$$\eta W^T(t) + W(t) \eta = 0 . \tag{4.11}$$

The Lax propagation equation is normalized as follows:

$$\frac{d}{dt} L(t) + [W(t), L(t)] = 0 . \quad (4.12)$$

Decomposed along the basis of coset generators K_i eq.(4.12) yields the following system of five differential equations:

$$\begin{aligned} -\frac{Y_3(t)^2}{\sqrt{2}} - \frac{Y_4(t)^2}{2\sqrt{2}} - \frac{Y_5(t)^2}{2\sqrt{2}} + \frac{d}{dt} Y_1(t) &= 0 , \\ -\frac{1}{2}\sqrt{\frac{3}{2}}Y_4(t)^2 + \frac{1}{2}\sqrt{\frac{3}{2}}Y_5(t)^2 + \frac{d}{dt} Y_2(t) &= 0 , \\ -\sqrt{2}Y_1(t)Y_3(t) - Y_4(t)Y_5(t) + \frac{d}{dt} Y_3(t) &= 0 , \\ \frac{Y_1(t)Y_4(t)}{\sqrt{2}} + \sqrt{\frac{3}{2}}Y_2(t)Y_4(t) - Y_3(t)Y_5(t) + \frac{d}{dt} Y_4(t) &= 0 , \\ -\frac{Y_1(t)Y_5(t)}{\sqrt{2}} + \sqrt{\frac{3}{2}}Y_2(t)Y_5(t) + \frac{d}{dt} Y_5(t) &= 0 . \end{aligned} \quad (4.13)$$

This is the first order differential system on the 5-dimensional Poissonian manifold provided by the Borel subalgebra of $B(\mathfrak{sl}(3)) \subset \mathfrak{sl}(3)$ which, as a consequence of the general discussion of section 2.2, is Liouville integrable. On this 5-dimensional manifold there are four hamiltonian functions in involution. Of these latter one is just zero, one is a Casimir, that labels the 4-dimensional symplectic leaves into which the 5-dimensional manifold foliates. The remaining two functions are the 2-hamiltonians which guarantee the Liouville integrability of each 4-dimensional symplectic leaf. In the next section we consider the explicit form of such hamiltonians.

4.3 The hamiltonian functions in involution

As recalled above the differential system (4.13) is Liouville integrable since it admits four hamiltonian functions in involution, that can be explicitly constructed according to formula (2.49). The first of these four hamiltonians is identically zero since it corresponds to the trace of the Lax operator, namely to the extra generator of $\mathfrak{gl}(3, \mathbb{R})$ which is deleted in order to step down to $\mathfrak{sl}(3, \mathbb{R})$. The last, which, instead of being polynomial, is rational, corresponds to the advertised Casimir labeling the leaves of the foliation.

Explicitly from (2.49) we obtain the following result:

$$\mathfrak{h}_1 \doteq \mathfrak{h}_{01} = 0 , \quad (4.14)$$

$$\mathfrak{h}_2 \doteq \mathfrak{h}_{02} = \frac{1}{2}Y_1(t)^2 + \frac{1}{2}Y_2(t)^2 - \frac{1}{4}Y_3(t)^2 + \frac{1}{4}Y_4(t)^2 - \frac{1}{4}Y_5(t)^2 , \quad (4.15)$$

$$\mathfrak{h}_3 \doteq \mathfrak{h}_{03} = \frac{Y_2(t)^3}{3\sqrt{6}} - \frac{Y_1(t)^2 Y_2(t)}{\sqrt{6}} + \frac{Y_3(t)^2 Y_2(t)}{2\sqrt{6}} + \frac{Y_4(t)^2 Y_2(t)}{4\sqrt{6}} - \frac{Y_5(t)^2 Y_2(t)}{4\sqrt{6}}$$

$$-\frac{Y_1(t)Y_4(t)^2}{4\sqrt{2}} - \frac{Y_1(t)Y_5(t)^2}{4\sqrt{2}} + \frac{1}{4}Y_3(t)Y_4(t)Y_5(t) , \quad (4.16)$$

$$\mathfrak{h}_4 \doteq \mathfrak{h}_{11} = \frac{Y_1(t)}{\sqrt{2}} + \frac{Y_2(t)}{\sqrt{6}} - \frac{Y_3(t)Y_4(t)}{2Y_5(t)} \quad (4.17)$$

and by explicit calculation we can verify that the functions \mathfrak{h}_A ($A = 1, \dots, 4$) are constant along the Toda flow, namely, upon use of eq.s(4.13) it is identically true that

$$\partial_t \mathfrak{h}_A = 0 . \quad (4.18)$$

If we calculate the secular equation for the Lax operator (4.9) we get

$$\begin{aligned} 0 &= \text{Det} (L(t) - \lambda \mathbf{1}) \\ &= -\lambda^3 + \mathfrak{h}_2 \lambda + \mathfrak{h}_3 . \end{aligned} \quad (4.19)$$

Hence parameterizing the Toda flows by means of the values of the hamiltonians, the eigenvalues of the Lax operator are given by the three roots of the cubic equation (4.19). Using Cardano's formula these three roots are given by

$$\begin{aligned} \lambda_1 &= -\frac{2\sqrt[3]{3}\mathfrak{h}_2 + \sqrt[3]{2}(\sqrt{\Delta} - 9\mathfrak{h}_3)^{2/3}}{6^{2/3}\sqrt[3]{\sqrt{\Delta} - 9\mathfrak{h}_3}} , \\ \lambda_2 &= \frac{2(3i + \sqrt{3})\mathfrak{h}_2 + \sqrt[3]{2}\sqrt[3]{3}(1 - i\sqrt{3})(\sqrt{\Delta} - 9\mathfrak{h}_3)^{2/3}}{22^{2/3}3^{5/6}\sqrt[3]{\sqrt{\Delta} - 9\mathfrak{h}_3}} , \\ \lambda_3 &= \frac{2(-3i + \sqrt{3})\mathfrak{h}_2 + \sqrt[3]{2}\sqrt[3]{3}(1 + i\sqrt{3})(\sqrt{\Delta} - 9\mathfrak{h}_3)^{2/3}}{22^{2/3}3^{5/6}\sqrt[3]{\sqrt{\Delta} - 9\mathfrak{h}_3}} \end{aligned} \quad (4.20)$$

where

$$\Delta \equiv -12\mathfrak{h}_2^3 + 81\mathfrak{h}_3^2 \quad (4.21)$$

is the discriminant of the cubic equation.

4.4 Normal form of the Lax operator

The discussion of the normal forms for the coset $\frac{\text{SL}(3;\mathbb{R})}{\text{SO}(1,2;\mathbb{R})}$ splits in two cases since, from the definition of the number p we have

$$p \equiv \text{rank} \left(\frac{\mathbf{H}^*}{\mathbf{H}_c} \right) = \text{rank} \left(\frac{\text{SO}(1,2;\mathbb{R})}{\text{SO}(2;\mathbb{R})} \right) = 1 . \quad (4.22)$$

Indeed the embedding (3.3) is trivial in this case

$$\begin{aligned} \mathbb{U} &\equiv \mathfrak{sl}(3, \mathbb{R}) \hookrightarrow \mathfrak{sl}(3, \mathbb{R}) , \\ \mathbb{U} \supset \mathbb{H}^* &\equiv \mathfrak{so}(1, 2; \mathbb{R}) \hookrightarrow \mathfrak{so}(1, 3-1; \mathbb{R}) \subset \mathfrak{sl}(3, \mathbb{R}) . \end{aligned} \quad (4.23)$$

The two possible spectral type are characterized by $k = 0$ or $k = 1$.

4.4.1 Spectral type $k = 0$

In this case we have from eq. (3.32) $\mathcal{T} = \widehat{T}_0 = \mathbf{1}_3$ and the normal form of the Lax operator is

$$\widehat{\Lambda}_0 = \Lambda_0 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1 - \lambda_2 \end{pmatrix} \quad (4.24)$$

where $\lambda_{1,2}$ are two real eigenvalues and $-\lambda_1 - \lambda_2$ is the third, also real. From the point of view of the Lie algebra, the normal form (4.24) is just a linear combination of the two Cartan generators $K_{1,2}$. Indeed we can write

$$\widehat{\Lambda} = \frac{1}{\sqrt{2}} (\lambda_1 - \lambda_2) K_1 - \sqrt{\frac{3}{2}} (\lambda_1 + \lambda_2) K_2. \quad (4.25)$$

A generic element of the $\text{SO}(1, 2; \mathbb{R})$ group can be parameterized as a product of three elements of the three one-parameter subgroups, namely we can set

$$\begin{aligned} \text{SO}(1, 2; \mathbb{R}) \ni \mathcal{O}(v, \theta, w) &= \exp[v J_1] \cdot \exp[\theta J_2] \cdot \exp[w J_3] = \\ &= \begin{pmatrix} \cosh(v) \cosh(w) - \sin(\theta) \sinh(v) \sinh(w) & -\cos(\theta) \sinh(v) & \cosh(w) \sin(\theta) \sinh(v) - \cosh(v) \sinh(w) \\ \cosh(v) \sin(\theta) \sinh(w) - \cosh(w) \sinh(v) & \cos(\theta) \cosh(v) & \sinh(v) \sinh(w) - \cosh(v) \cosh(w) \sin(\theta) \\ -\cos(\theta) \sinh(w) & \sin(\theta) & \cos(\theta) \cosh(w) \end{pmatrix} \end{aligned} \quad (4.26)$$

where the parameters v, w and θ can be regarded as the three Euler angles (two hyperbolic and one elliptic) that parameterize $\text{SO}(1, 2; \mathbb{R})$. Hence in the spectral type $k = 0$ the initial value of the Lax operator at *time* $t = 0$ can be written as

$$L_0(0) = \mathcal{O}_0 \cdot \widehat{\Lambda}_0 \cdot \mathcal{O}_0^{-1} \quad (4.27)$$

where

$$\mathcal{O}_0 \equiv \mathcal{O}(v, \theta, w). \quad (4.28)$$

In this way the matrix $L_0(0)$ depends on the five real parameters $\{\lambda_1, \lambda_2, v, \theta, w\}$ which parameterize the initial conditions $Y_i(0)$ for the five real fields $Y_i(t)$. Indeed the values $Y_i(0)$ as functions of $\{\lambda_1, \lambda_2, v, \theta, w\}$ can be extracted by projecting $L_0(0)$ along the orthonormal basis of coset generators K_i .

4.4.2 Spectral type $k = 1$

In this case we have from eq. (3.32)

$$\mathcal{T} = \widehat{T}_1 \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.29)$$

and the eigenvalues of the Lax operator are given by a pair of complex conjugate eigenvalues $\lambda_1 = x + iy$, $\lambda_2 = x - iy$, while the third one is the negative of their sum, $\lambda_3 = -2x$. Hence the normal form of the Lax operator is as follows:

$$\begin{aligned}\widehat{\Lambda}_1 &= \mathcal{T} \cdot \Lambda_1 \cdot \mathcal{T}^{-1} \equiv \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & -2x \end{pmatrix}, \\ \Lambda_1 &= \begin{pmatrix} x + iy & 0 & 0 \\ 0 & x - iy & 0 \\ 0 & 0 & -2x \end{pmatrix}.\end{aligned}\tag{4.30}$$

As element of the Lie algebra, rather than being a linear combination of the two Cartan generators, in this case, the normal form is a linear combination of one Cartan and one of the compact coset generators. Indeed we have

$$\widehat{\Lambda}_1 = -\sqrt{6}x K_2 - \sqrt{2}y K_3.\tag{4.31}$$

Hence in the spectral type $k = 1$ the initial value of the Lax operator at *time* $t = 0$ is written as

$$L_1(0) = \mathcal{O}_0 \cdot \widehat{\Lambda}_1 \cdot \mathcal{O}_0^{-1}\tag{4.32}$$

the rotation matrix \mathcal{O}_0 being defined in eq.(4.28). As a result the initial values $Y_i(0)$ of the 5 fields are now parameterized by the five parameters $\{x, y, v, \theta, w\}$.

In both spectral types we can calculate the values of the constant hamiltonians \mathfrak{h}_A as functions of the five real parameter set, either $\{\lambda_1, \lambda_2, v, \theta, w\}$ or $\{x, y, v, \theta, w\}$. This is what we do in the next section.

4.5 Characterization of orbits through the hamiltonians

It is now instructive to characterize the orbits and hence the normal forms of the Lax operators through the values of the conserved hamiltonians responsible for the integrability of the system.

As we have seen the two hamiltonians entering the secular equation and hence the determination of the eigenvalues are $\mathfrak{h}_{2,3}$. Furthermore, what distinguishes the two spectral types is the sign of the discriminant Δ defined in equation (4.21). When $\Delta < 0$ we have three real eigenvalues, while when $\Delta > 0$ we have a pair of complex conjugate eigenvalues and a third real one. The regions in the $\mathfrak{h}_2, \mathfrak{h}_3$ plane corresponding to the two spectral types are visualized in Fig.1 It is now instructive to evaluate the explicit form of the hamiltonians and of the discriminant in the two spectral types.

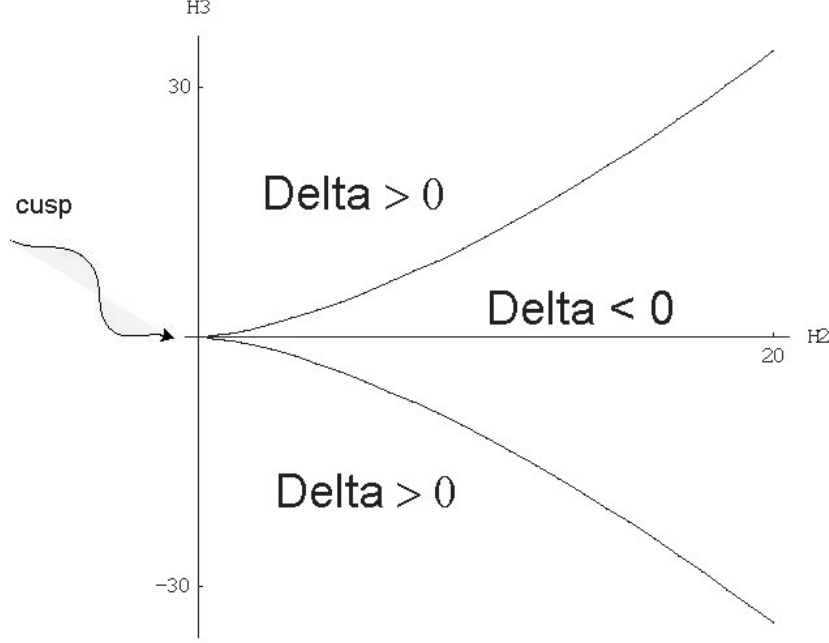


Figure 1: Phase-diagram in the $\mathfrak{h}_2, \mathfrak{h}_3$ plane. In the locus $\Delta > 0$ we have complex conjugate eigenvalues and spectral type $k = 1$. In the locus $\Delta < 0$ there are three real eigenvalues and spectral type $k = 0$. The line $\Delta = 0$ which separates the two regions is a singular locus where two eigenvalues coincide and we have an enhancement of symmetry. On this locus the normal form admits a one-parameter stability subgroup. The cuspidal point $\mathfrak{h}_2 = \mathfrak{h}_3 = 0$ corresponds to nilpotent Lax operators as we discuss in the paper [36] submitted to the hep-th arXiv after the first appearance of the present paper.

4.5.1 The hamiltonians in spectral type $k = 0$

From the definition given in eq.s(4.14-4.17), by using the initial Lax operator (4.27) to calculate the fields $Y_i(0)$ and hence the hamiltonians, we find

$$\mathfrak{h}_1 = 0, \quad (4.33)$$

$$\mathfrak{h}_2 = \lambda_1^2 + \lambda_2 \lambda_1 + \lambda_2^2, \quad (4.34)$$

$$\mathfrak{h}_3 = -\lambda_1 \lambda_2 (\lambda_1 + \lambda_2), \quad (4.35)$$

$$\begin{aligned} \mathfrak{h}_4 = & - \left[2 (\sin \theta \sinh v \lambda_1^2 + 2 \cosh w (\cosh w \sin \theta \sinh v - \cosh v \sinh w) \lambda_2 \lambda_1 \right. \\ & \left. + \sinh w (\sin \theta \sinh v \sinh w - \cosh v \cosh w) \lambda_2^2 \right] \\ & \times \left[2 (\sin \theta \sinh v \cosh^2 w - 2 \cosh v \sinh w \cosh w + \sin \theta \sinh v \sinh^2 w) \lambda_1 \right. \\ & \left. + ((\cosh 2w + 3) \sin \theta \sinh v - 2 \cosh v \cosh w \sinh w) \lambda_2 \right]^{-1}, \end{aligned} \quad (4.36)$$

$$\Delta = -3 (\lambda_1 - \lambda_2)^2 (2\lambda_1 + \lambda_2)^2 (\lambda_1 + 2\lambda_2)^2. \quad (4.37)$$

As it is evident from the above explicit expressions, in the spectral type $k = 0$ the discriminant is strictly negative and it reaches the value zero only in the case of degenerate eigenvalues, namely when any two of the three eigenvalues are equal.

4.5.2 The hamiltonians in the spectral type $k = 1$

From the definition given in eq.s(4.14-4.17), by using the initial Lax operator (4.32) to calculate the fields $Y_i(0)$ and hence the hamiltonians, we find

$$\mathfrak{h}_1 = 0 , \quad (4.38)$$

$$\mathfrak{h}_2 = 3x^2 - y^2 , \quad (4.39)$$

$$\mathfrak{h}_3 = -2x(x^2 + y^2) , \quad (4.40)$$

$$\begin{aligned} \mathfrak{h}_4 = & \left[\cosh v \left(\cos \theta \sinh 2wy^2 + 2x \cosh w (2y \sin \theta + 3x \cos \theta \sinh w) \right) - \right. \\ & \sinh v \left((3x^2 + y^2) \cos \theta \sin \theta \cosh^2 w + (3x^2 + y^2) \cos \theta \sin \theta \sinh^2 w \right. \\ & \left. \left. + (3x^2 + y^2) \cos \theta \sin \theta - 4xy \cos 2\theta \sinh w \right) \right] \times \\ & \times \left[2 \left(\frac{3}{2} x \cos \theta \sin \theta \sinh v \cosh^2 w + y \cosh v \sin \theta \cosh w \right. \right. \\ & \left. \left. - 3x \cos \theta \cosh v \sinh w \cosh w + \frac{3}{2} x \cos \theta \sin \theta \sinh v \sinh^2 w \right. \right. \\ & \left. \left. + \frac{3}{2} x \cos \theta \sin \theta \sinh v + y \cos(2\theta) \sinh v \sinh w \right) \right]^{-1} , \end{aligned} \quad (4.41)$$

$$\Delta = 12y^2 (9x^2 + y^2)^2 . \quad (4.42)$$

Once again also in this branch of the solution space the sign of the discriminant is definite. Δ is positive definite and it vanishes only for $y = 0$. This condition however corresponds to real degenerate eigenvalues and matches the same condition obtained from the other branch with spectral type $k = 0$.

5 Examples of explicit solutions

In this section we illustrate the integration algorithm by considering some example of solutions corresponding to the three spectral types: $k = 0$, $k = 1$ and degenerate.

5.1 An example of solution of spectral type $k = 0$

A very simple solution of this spectral type can be obtained fixing the following initial data:

$$\{\lambda_1, \lambda_2, \lambda_3\} = \left\{ \frac{1}{2}, \frac{3}{2}, -2 \right\} , \quad (5.1)$$

$$\mathcal{O}_0 = \exp[2 J_1] = \begin{pmatrix} \frac{1+e^4}{2e^2} & -\frac{-1+e^4}{2e^2} & 0 \\ -\frac{-1+e^4}{2e^2} & \frac{1+e^4}{2e^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (v=2, \theta=0, w=0). \quad (5.2)$$

With these data the initial form of the Lax operator is the following:

$$L_0(0) = O_0 \cdot \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & -2 \end{pmatrix} \cdot O_0^{-1} = \begin{pmatrix} 1 - \frac{1}{4e^4} - \frac{e^4}{4} & -\frac{-1+e^8}{4e^4} & 0 \\ \frac{-1+e^8}{4e^4} & \frac{1}{4} \left(4 + \frac{1}{e^4} + e^4\right) & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (5.3)$$

As we see $L_0(0)$ has a block diagonal structure $2+1$. Such a block structure is preserved throughout the all flow from $t = -\infty$ to $t = +\infty$ as we can deduce from the explicit result of the integration

$$Y_1(t) = -\frac{1 - 2e^4 + e^8 + e^{2t} + 2e^{2t+4} + e^{2t+8}}{\sqrt{2}(-1 + 2e^4 - e^8 + e^{2t} + 2e^{2t+4} + e^{2t+8})}, \quad (5.4)$$

$$Y_2(t) = -\sqrt{6}, \quad (5.5)$$

$$Y_3(t) = -\frac{2e^t(-1 + e^8)}{-1 + 2e^4 - e^8 + e^{2t} + 2e^{2t+4} + e^{2t+8}}, \quad (5.6)$$

$$Y_4(t) = 0, \quad (5.7)$$

$$Y_5(t) = 0. \quad (5.8)$$

The vanishing of both $Y_4(t)$ and $Y_5(t)$ is what guarantees the block diagonal structure of the Lax operator. The same fact however implies that the 4-th hamiltonian, the rational one is indeterminate in this case, being the ratio of two zeros. The other two (polynomial) hamiltonians have instead the following explicit values:

$$\mathfrak{h}_2 = \frac{13}{4} \quad ; \quad \mathfrak{h}_3 = -\frac{3}{2}. \quad (5.9)$$

The plot of the two non-trivial functions $Y_{1,3}(t)$ is exhibited in Fig.2. As we see there is a singularity in both fields at a finite time $t = t_0 \simeq -0.03663$. This singularity separates the range of the variable t in two parts. We can consider the solution only on one side of the singularity. Let us consider for instance the Cartan field $Y_1(t)$. As we know this function is actually the derivative of the corresponding Cartan field $h_1(t)$ [10] and in order to reconstruct the physical interpretation of our solution we are supposed to perform a second integration

$$h_1(t) = \int Y_1(t) dt = \frac{t - \log|-1 + 2e^4 - e^8 + e^{2t} + 2e^{2t+4} + e^{2t+8}|}{\sqrt{2}}. \quad (5.10)$$

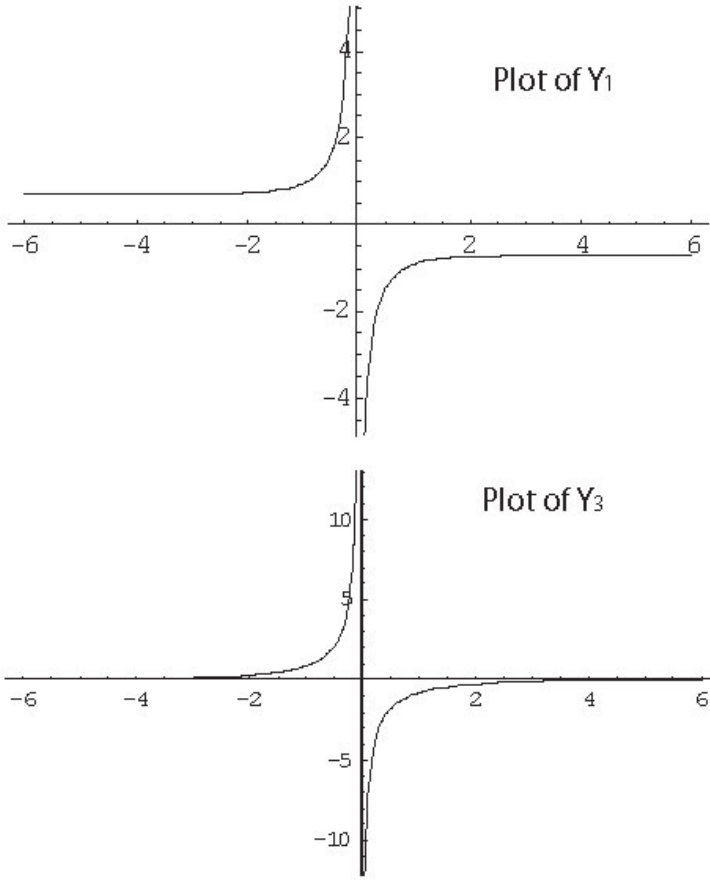


Figure 2: Plot of the $Y_{1,3}(t)$ fields in the solution of spectral type $k = 0$ characterized by the following initial data: $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{3}{2}$, $v = 2$, $\theta = w = 0$. In both fields there is a singularity at $t = t_s \simeq -0.03663$. The other three fields are constant or even zero.

The singularity at a finite time is a qualitative difference between the type of solutions encountered in the pseudo-Riemannian case and those encountered in the case of cosmic billiards (Riemannian coset manifolds). A similarity, instead, which exists between the spectral type $k = 0$ of pseudo-Riemannian system and the billiard case is the asymptotic behavior at $t = \pm\infty$. For this spectral type (but not for the other), just as in the billiard case, the Lax operator tends asymptotically to a diagonal form which differs from Λ only by a permutation of the eigenvalues. We can verify this statement in the present example. We find

$$L_0(-\infty) = \begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -2 \end{pmatrix} \neq \Lambda ,$$

$$L_0(\infty) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & -2 \end{pmatrix} = \Lambda. \quad (5.11)$$

5.2 Another solution of the spectral type $k = 0$ with finite hamiltonians

To appreciate the differences we apply the integration algorithm to the case where the choice of the eigenvalues and of the spectral type remains the same as in the previous example but we modify the initial rotation element \mathcal{O}_0 by switching on also a compact rotation angle $\theta = \frac{\pi}{4}$. So we set

$$\mathcal{O}_0 = \exp[2J_1] \cdot \exp\left[\frac{\pi}{4}J_2\right] = \begin{pmatrix} \frac{1+e^4}{2e^2} & -\frac{-1+e^4}{2\sqrt{2}e^2} & \frac{-1+e^4}{2\sqrt{2}e^2} \\ -\frac{-1+e^4}{2e^2} & \frac{1+e^4}{2\sqrt{2}e^2} & -\frac{1+e^4}{2\sqrt{2}e^2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (5.12)$$

and for the initial Lax operator we get

$$L_0(0) = \mathcal{O}_0 \Lambda \mathcal{O}_0^{-1} = \begin{pmatrix} \frac{3+2e^4+3e^8}{16e^4} & \frac{3(-1+e^8)}{16e^4} & -\frac{7(-1+e^4)}{8e^2} \\ -\frac{3(-1+e^8)}{16e^4} & -\frac{3-2e^4+3e^8}{16e^4} & \frac{7(1+e^4)}{8e^2} \\ \frac{7(-1+e^4)}{8e^2} & \frac{7(1+e^4)}{8e^2} & -\frac{1}{4} \end{pmatrix}. \quad (5.13)$$

Calculating the hamiltonians from the above form of the initial Lax operator we obtain

$$\{\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4\} = \left\{0, \frac{13}{4}, -\frac{3}{2}, -\frac{1}{2}\right\}. \quad (5.14)$$

As we see \mathfrak{h}_2 and \mathfrak{h}_3 , which depend only on the eigenvalues are the same as before. On the other hand, \mathfrak{h}_4 is no longer undefined as in the previous case and obtains the finite rational value $-\frac{1}{2}$. This is so because the new initial value of the Lax operator as no degenerate minors has in the previous case.

The new explicit solution is given by the following functions:

$$Y_1(t) = -\frac{(-2+5e^{7t})(1-2e^4+e^8+2e^{2t}+e^{7t}+4e^{2t+4}+2e^{2t+8}-2e^{7t+4}+e^{7t+8})}{2\sqrt{2}(1+e^{7t})(1-2e^4+e^8-2e^{2t}+e^{7t}-4e^{2t+4}-2e^{2t+8}-2e^{7t+4}+e^{7t+8})}, \quad (5.15)$$

$$Y_2(t) = \frac{\sqrt{\frac{3}{2}}(-4+3e^{7t})}{2(1+e^{7t})}, \quad (5.16)$$

$$Y_3(t) = -\frac{\sqrt{2}e^t(-1+e^8)(-2+5e^{7t})}{\sqrt{1+e^{7t}}(1-2e^4+e^8-2e^{2t}+e^{7t}-4e^{2t+4}-2e^{2t+8}-2e^{7t+4}+e^{7t+8})}, \quad (5.17)$$

$$Y_4(t) = -\frac{7\sqrt{2}e^{-t-2}(1+e^4)}{\sqrt{-e^{-11t-4}(1+e^{7t})^2(1-2e^4+e^8-2e^{2t}+e^{7t}-4e^{2t+4}-2e^{2t+8}-2e^{7t+4}+e^{7t+8})}}, \quad (5.18)$$

$$Y_5(t) = -\frac{7(-1+e^4)}{e^2\sqrt{-e^{-7t-4}(1+e^{7t})(1-2e^4+e^8-2e^{2t}+e^{7t}-4e^{2t+4}-2e^{2t+8}-2e^{7t+4}+e^{7t+8})}}. \quad (5.19)$$

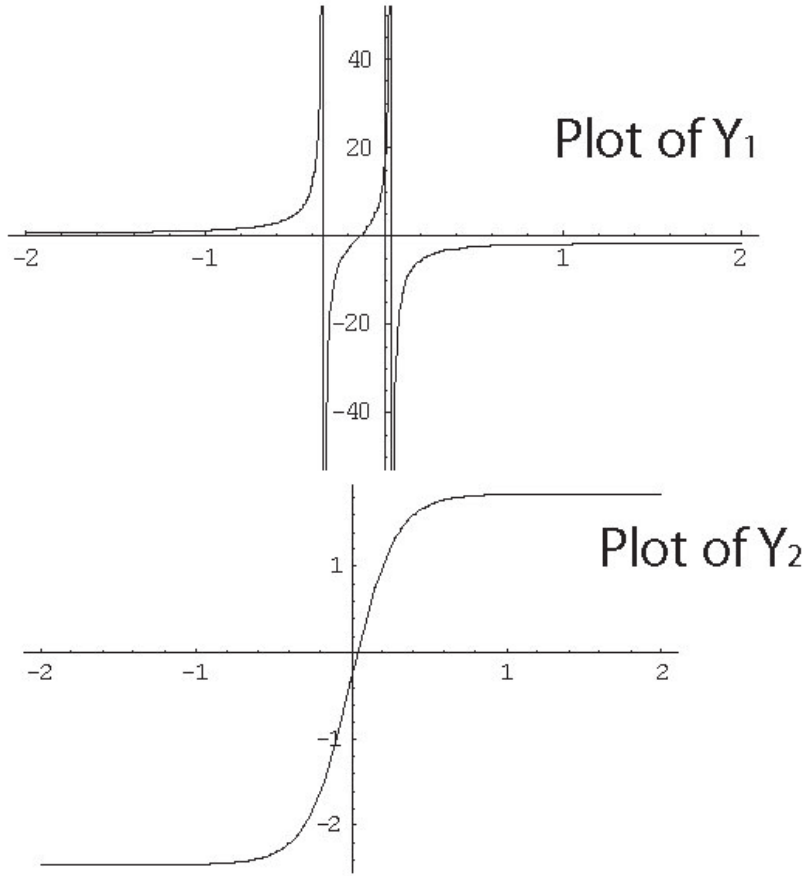


Figure 3: Plot of the Cartan fields $Y_{1,2}(t)$ in the solution of spectral type $k = 0$ characterized by the following initial data: $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{3}{2}$, $v = 2$, $\theta = \pi/4$, $w = 0$. In $Y_1(t)$ there are two singularities at finite times $t = t_{s_1} \simeq -0.338507$ and $t_{s_2} \simeq 0.0417536$.

The behavior of the solution is qualitatively similar to that discussed in the previous case. This is evident from the plots of the Cartan fields exhibited in Fig.3. As one realizes there are just two singularities at finite time $t = t_{s_1} \simeq -0.338507$ and $t_{s_2} \simeq 0.0417536$, that affect one of the two Cartans but not the other. The same singularities appear in the field $Y_3(t)$. The other two fields $Y_{4,5}(t)$ are real only in the interval between the two singularities as it evident from the plot of, for instance, $Y_4(t)$, exhibited in Fig.4. This means that the overall solution is properly defined only in the interval between the two singularities. Notwithstanding this fact if we calculate the asymptotic limit of the Lax

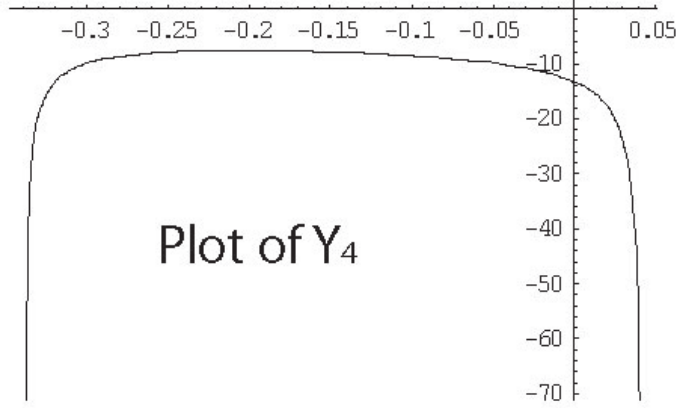


Figure 4: Plot of the non-compact nilpotent fields $Y_4(t)$ in the solution of spectral type $k = 0$ characterized by the following initial data: $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{3}{2}$, $v = 2$, $\theta = \pi/4$, $w = 0$. The function $Y_4(t)$ is real only in the interval comprised between the two singularities $t = t_{s_1} \simeq -0.338507$ and $t_{s_2} \simeq 0.0417536$.

operator at $\pm\infty$ we obtain finite diagonal real forms. Indeed we find

$$\begin{aligned} L_0(-\infty) &= \begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -2 \end{pmatrix} \neq \Lambda , \\ L_0(\infty) &= \begin{pmatrix} -2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix} \neq \Lambda . \end{aligned} \quad (5.20)$$

In the present example the difference between the order of eigenvalues at $+\infty$ and at $-\infty$ is provided by the permutation of highest order, just as it happens in the billiard Riemannian case for flows not touching singular surfaces. The previous case did not have this property because it developed on a singular surface and the indeterminacy of the fourth hamiltonian was a sign of that.

Although similar to the billiard case the asymptotic limits loose their meaningfulness in the pseudo-Riemannian case since they are separated from the physical flow region by regions where the Lax operator becomes complex. The real asymptotic diagonal limits are approached through imaginary values. The physical flow region is typically bounded by singularities.

5.3 An example of spectral type $k = 1$

As an example of the other spectral type we choose the solution generated by the following very simple initial data:

$$k = 1 \quad x = 1 \quad ; \quad y = 1 , \quad (5.21)$$

$$v = 0 , \quad (5.22)$$

$$\theta = \frac{\pi}{3} , \quad (5.23)$$

$$w = 0 . \quad (5.24)$$

The corresponding rotation matrix is

$$\mathcal{O}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad (5.25)$$

and the resulting initial Lax operator is

$$L_1(0) = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{2} & -\frac{5}{4} & \frac{3\sqrt{3}}{4} \\ \frac{\sqrt{3}}{2} & \frac{3\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix} . \quad (5.26)$$

The resulting vector of hamiltonians is

$$\{\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4\} = \{0, 2, -4, 2\} \quad (5.27)$$

and the explicit form of the solution is given by

$$Y_1(t) = \frac{\sec(2t) (9e^{6t} \cos(4t) + 4 \sin(2t) + 3e^{6t}(\sin(4t) + 3))}{\sqrt{2} (6e^{6t} \cos(2t) + 2)} , \quad (5.28)$$

$$Y_2(t) = \frac{\sqrt{\frac{3}{2}} (3e^{6t} \cos(2t) - 3e^{6t} \sin(2t) - 2)}{3e^{6t} \cos(2t) + 1} , \quad (5.29)$$

$$Y_3(t) = -\frac{2\sqrt{2}e^{-3t}}{\cos^{\frac{1}{2}}(2t) \sqrt{2e^{-6t} \cos(2t) + 3 \cos(4t) + 3}} , \quad (5.30)$$

$$Y_4(t) = -\frac{2\sqrt{6}e^{-2t} (3 \cos(2t) - \sin(2t))}{\sqrt{3e^{2t} \cos(2t) + e^{-4t}} \sqrt{2e^{-6t} \cos(2t) + 3 \cos(4t) + 3}} , \quad (5.31)$$

$$Y_5(t) = -\frac{2\sqrt{3}e^t}{\cos^{\frac{1}{2}}(2t) \sqrt{3e^{2t} \cos(2t) + e^{-4t}}} . \quad (5.32)$$

The structure of this solution can be considered analyzing the plots of the various fields. The Cartan fields exhibit a quasi periodic behavior (with singularities) displayed in fig.5.

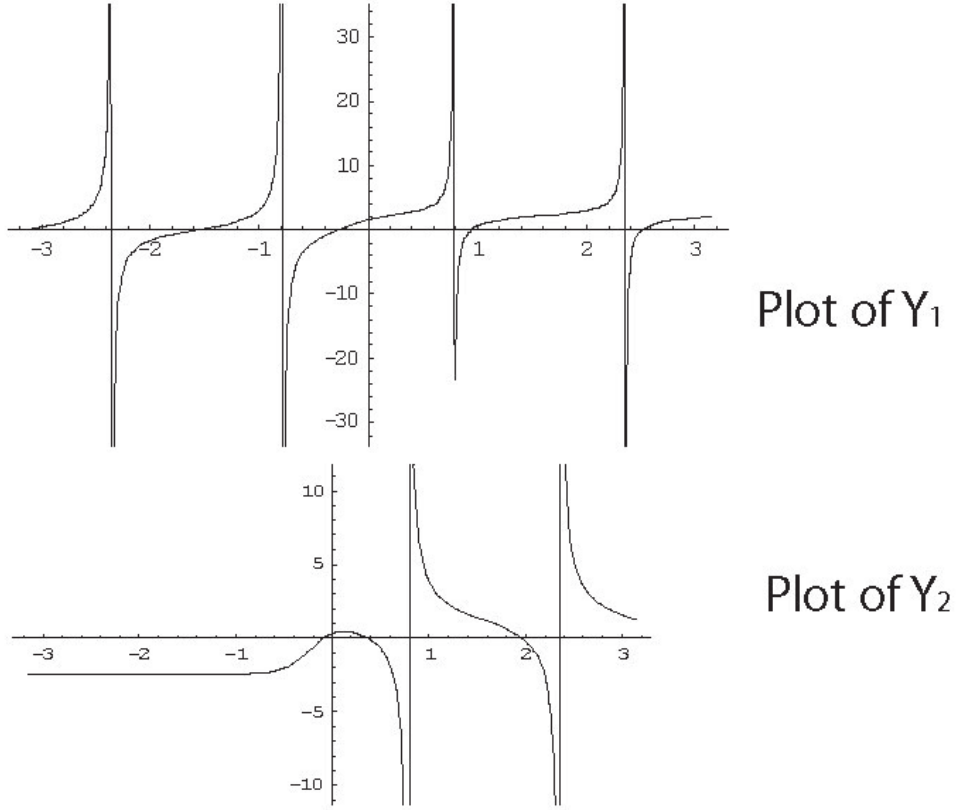


Figure 5: Plot of the two Cartan fields $Y_{1,2}(t)$ in the solution of spectral type $k = 1$, with parameters $x = 1, y = 1, v = 0, \theta = \frac{\pi}{3}, w = 0$.

The two fields $Y_{3,5}(t)$ have instead a periodic real behavior for $t > t_m$ and for $t < t_p$ respectively, where

$$t_p \simeq 0.785398 \quad ; \quad t_m \simeq -0.785398 = -t_p \quad (5.33)$$

are finite times. Respectively below and above these singularity barriers the fields $Y_{3,5}(t)$ become imaginary. This is evident from the plots displayed in Fig. 6 This behavior restricts the physical range of the solution to the interval $[-t_p, t_p]$. This is further confirmed by the plot of the field $Y_4(t)$ which is real only in the same interval. This is seen in Fig.7

In this way we come to the conclusion that, notwithstanding the appearance of periodic functions and the periodic behavior of some of the fields of the system, also in the case of the spectral type $k = 1$, the generic form of the real solution appears to be the evolution on a finite range of the *time*-line, bounded at the extrema by singularities. A similar generic behavior is suitable for the description of an evolution from spatial infinity to a horizon as it happens in black hole physics.

A detailed study of the solution space, a classification of the asymptotic limits and

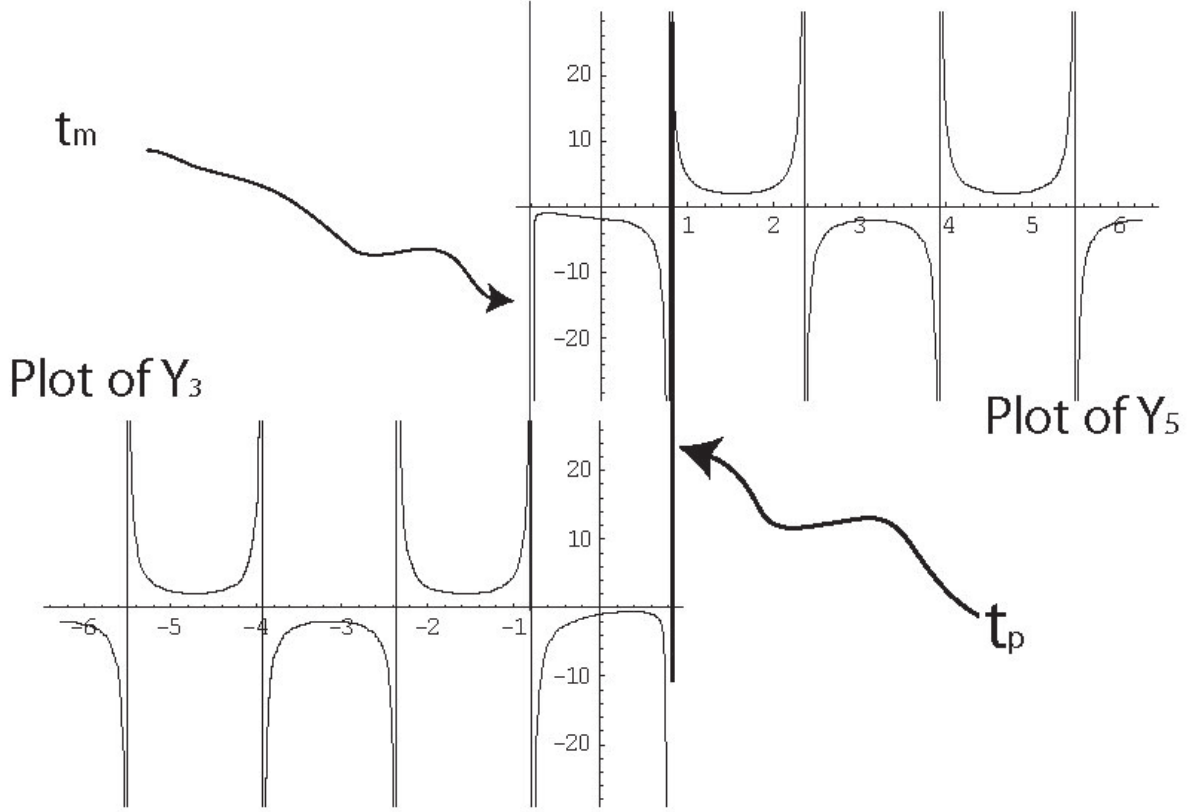


Figure 6: Plot of the non Cartan fields $Y_{3,5}(t)$ in the solution of spectral type $k = 1$, with parameters $x = 1, y = 1, v = 0, \theta = \frac{\pi}{3}, w = 0$.

the analysis of critical surfaces in the moduli space is postponed to future publications, where the physical interpretation of the Lax equation solutions in connection with p -brane physics will be addressed.

6 Conclusions

In this paper we have presented a new view-point on the integrability of supergravity cosmic billiards and black holes that is based on the Poissonian structure of the underlying solvable Lie algebra \mathcal{S} .

The main results of our paper are two:

- The explicit construction of the integration algorithm extended also to the case of Lorentzian cosets U/H^* .
- The explicit construction of the hamiltonian functions in involution \mathfrak{h}_α responsible for Liouville integrability.

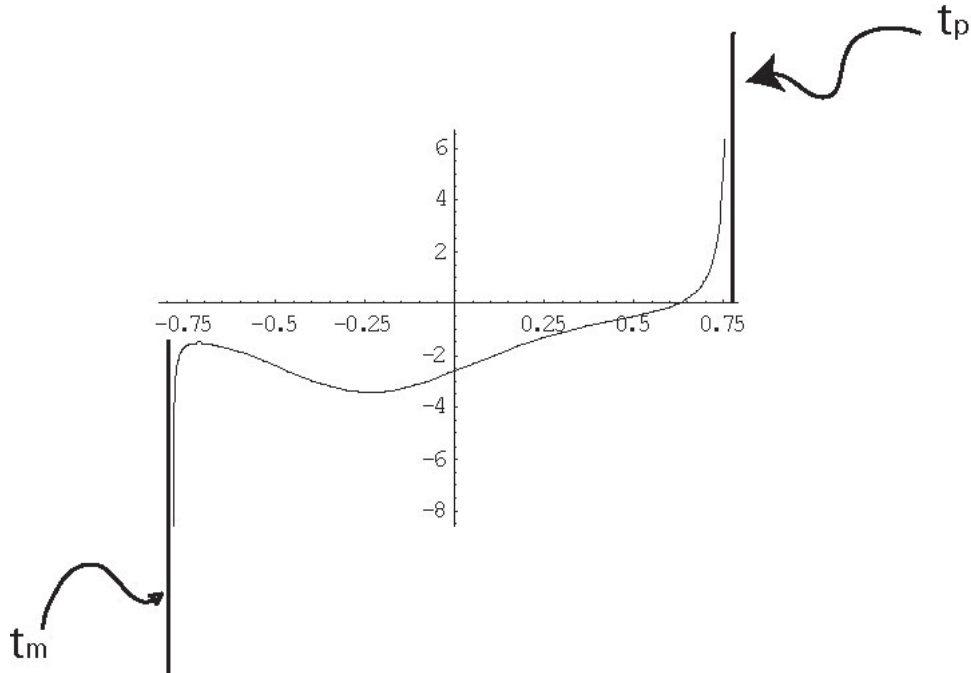


Figure 7: Plot of the non Cartan field $Y_4(t)$ in the solution of spectral type $k = 1$, with parameters $x = 1, y = 1, v = 0, \theta = \frac{\pi}{3}, w = 0$. Outside of the plotted range the field is imaginary.

We believe that a systematic use of our techniques for the construction of black-hole and billiard solutions will provide new results and new insight. In particular the relation of the Hamiltonians and the Casimirs with the physical invariants of the solution, like the entropy or the total mass will prove very helpful and inspiring. We leave this to future coming publications.

A point which we have not yet addressed but which is of the highest relevance concerns the issue of global topology of the solution space. The solvable parametrization covers only open branches of this space and the question of how to glue together different branches is very important.

Also this issue is left over for future publications.

Acknowledgments We would like to express our gratitude to our frequent collaborator and excellent friend Mario Trigiante for the exchange of useful information we had with him at the end of this project. He attracted our attention to the classification of normal forms already performed in [16], which coincides with the classification of spectral types we came up with in our adaptation of the Kodama integration algorithm. Furthermore we have been informed that explicit solutions for the $SL(3, \mathbb{R})/SO(1, 2)$ case, although not yet published, were already constructed more than a month ago by him and his collaborators,

implementing their own adaptation of the Kodama algorithm. Although we have not yet seen these solutions we are absolutely confident that they will be coherent with our own results. It is our pleasure to acknowledge this fact publicly.

Note added in the revised version As stated in the previous acknowledgements, the authors of [35] had indeed independently adapted Kodama integration algorithm to the case of G/H^* Lax equations and had derived some explicit solutions for the cases of $SL(2, \mathbb{R})/SO(1, 1)$ and $SL(3, \mathbb{R})/SO(1, 2)$. The construction of the integration algorithm for the case of nilpotent initial Lax operators, which is relevant for extremal Black Holes, was performed in full generality in our paper [36]. In a revised version of their paper, which appeared the same day as our [36], the authors of [35] presented some particular solutions corresponding to specific nilpotent Lax operators pertaining to $SL(3, \mathbb{R})/SO(1, 2)$.

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